THE APRIL MEETING OF THE CHICAGO SECTION.

THE thirtieth regular meeting of the Chicago Section of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, April 5–6, 1912, extending through four half-day sessions. The total attendance was seventy-five, including the following fifty-six members of the Society:

Mr. Charles Ammerman, Professor R. P. Baker, Professor G. N. Bauer, Dr. E. G. Bill, Professor G. A. Bliss, Dr. R. L. Börger, Professor W. H. Bussey, Professor R. D. Carmichael, Professor H. E. Cobb, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor S. C. Davisson, Mr. W. W. Denton, Dr. Arnold Dresden, Professor L. E. Dickson, Professor W. B. Ford, Dr. T. H. Gronwall, Professor A. G. Hall, Professor E. R. Hedrick, Dr. T. H. Hildebrandt, Professor O. D. Kellogg, Professor A. M. Kenyon, Professor Kurt Laves, Professor A. C. Lunn, Dr. G. F. McEwen, Professor Malcolm McNeill, Dr. W. D. MacMillan, Professor G. A. Miller, Professor E. H. Moore, Professor F. M. Morrison, Professor F. R. Moulton, Mr. E. J. Moulton, Professor Alexander Pell, Professor W. D. Pemberton, Professor H. L. Rietz, Professor W. J. Risley, Professor W. H. Roever, Miss Ida M. Schottenfels, Mr. A. R. Schweitzer, Mr. T. M. Simpson, Professor C. H. Sisam, Professor E. B. Skinner, Professor H. E. Slaught, Mr. V. M. Spunar, Dr. E. B. Stouffer, Professor E. J. Townsend, Professor A. L. Underhill, Professor E. B. Van Vleck, Dr. G. E. Wahlin, Mr. C. W. Wester, Professor J. K. Whittemore, Professor E. J. Wilczynski, Professor R. E. Wilson, Professor A. E. Young, Professor J. W. A. Young, Professor Alexander Ziwet.

Professor D. R. Curtiss, chairman of the Section, presided throughout the four sessions, except during the presentation of his own paper, when Professor E. B. Van Vleck occupied the chair.

On Friday evening about fifty members dined together in the café of the University Commons and enjoyed another of those seasons of social intercourse to which all look forward with increasing interest.

At the business meeting on Saturday morning attention was called to the fact that the next meeting of the Amer-

ican Association for the Advancement of Science is to be at Cleveland, Ohio, in December, 1912, and that this occasion will afford an excellent opportunity for holding a general meeting of the American Mathematical Society at the same time and place. In this connection the following resolution was unanimously adopted: That the members of the Society in attendance at the April meeting of the Chicago Section respectfully request the Council to consider the advisability of holding the next annual meeting of the Society in Cleveland in affiliation with the American Association.

Under the head of "informal notes and queries," which was introduced into the programme in order to give the members an opportunity for the interchange of mathematical suggestions not related directly to any of the papers, there was a variety of interesting discussions participated in by Professors Moore, Hedrick, Curtiss, Dickson, Baker, Bliss, Moulton, Lunn, Dr. Bill, Dr. Gronwall, and Miss Schottenfels.

The following papers were presented at this meeting:

(1) Dr. S. LEFSCHETZ: "On birational transformations of three-space related to four-space varieties."

(2) Professor W. H. ROEVER: "Optical interpretations in higher geodesy."

(3) Professor W. H. ROEVER: "Mechanisms for illustrating lines of force."

(4) Professor W. H. ROEVER: "Deviations of falling bodies for a distribution not of revolution. Second paper."

(5) Professor ARNOLD EMCH: "The geometry of conformal rational transformations in a plane."

(6) Professor KURT LAVES: "The present state of the theory of Jupiter's five minor satellites."

(7) Professor D. R. CURTISS: "An extension of Descartes' rule of signs. Second paper."

(8) Professor J. K. WHITTEMORE: "Equality in geometry."

(9) Professor G. O. JAMES: "On the relation between the empirical and the inertial trihedrons of gravitational astronomy."

(10) Professor E. J. WILCZYNSKI: "A forgotten theorem of Newton on planetary motion and an instrumental solution of Kepler's equation."

(11) Dr. G. E. WAHLIN: "The relative number field $K(\sqrt[i]{a})$."

(12) Dr. E. G. BILL: "Analytic curves in non-euclidean space. Second paper."

(13) Professor L. C. KARPINSKI: "Algebra in the Quadripartitum numerorum of Johannes de Muris."

(14) Professor G. A. MILLER: "Infinite systems of indivisible groups."

(15) Mr. W. W. DENTON: "Projective differential geometry of developable surfaces."

(16) Professor R. P. BAKER: "The method of monodromy and its application to three-parameter quartics."

(17) Professor R. D. CARMICHAEL: "On transcendentally transcendental functions."

(18) Professor R. D. CARMICHAEL: "On the theory of linear difference equations."

(19) Professor J. W. GLOVER: "A general formula for the valuation of securities."

(20) Dr. E. L. DODD: "On ordinary plane and skew curves."

(21) Professor FLORIAN CAJORI: "On the Spanish symbol U for thousands."

(22) Mr. E. W. CHITTENDEN: "Infinite developments and the composition property $(K_{12}B_1)_*$ in general analysis."

(23) Mr. V. M. SPUNAR: "Note on Mersenne's numbers."

(24) Dr. E. B. STOUFFER: "The projective differential geometry of three-spreads generated by ∞^1 planes in five-dimensional space."

(25) Professor E. H. MOORE: "Multiplicative interrelations of certain classes of sequences of positive terms."

(26) Dr. T. H. GRONWALL: "On a theorem of Fejér's and an analogon to Gibbs's phenomenon."

(27) Dr. T. H. GRONWALL: "Some asymptotic expressions in the theory of numbers."

(28) Professor F. R. MOULTON: "Closed orbits of ejection and related periodic orbits in the problem of three bodies."

(29) Dr. T. H. HILDEBRANDT: "Necessary and sufficient conditions for the interchange of limit and summation for a special type of series."

Mr. Chittenden's paper was communicated to the Society through Professor Moore. In the absence of the authors, the papers of Dr. Lefschetz, Professor Emch, Professor James, Professor Karpinski, Professor Glover, Dr. Dodd, Professor Cajori, and Mr. Spunar, were read by title. Abstracts of the papers, except that of Mr. Spunar, follow below in the order of the titles above.

1. It is known that from two depictions of a rational surface we can obtain a birational transformation between two planes, which however is only a product of several quadratic ones. The same holds true for rational four-space varieties, except that the transformations obtained are not in general the product of simpler ones. In this paper Dr. Lefschetz considers the transformations deduced from depictions of quadratic and cubic varieties. The V_3^2 give the well known quadri-quadric transformations, having for base a conic and a point in each space. Several resulting transformations are considered, the simplest being related to a V_3^3 having two conic nodes. This is obtained by projecting the first space on the variety from one of the nodes, and then reprojecting from the other on the second space; it is a cubo-cubic transformation, the fundamental elements being in both spaces a sextic intersection of a cubic and a quadric surface and a point on it. This transformation is then applied to the depiction of a sextic surface having the fundamental sextic for nodal curve and the fundamental point for quadruple point with an elliptic quartic tangent cone, which is obtained from a quadric, and has, as can also be easily verified, $p_g = p_a = P = 0$, which are the conditions of rationality of a surface as given by Castelnuovo.

2. At a point P_0 of the rotating earth, the vertical is the straight line which coincides with a plumb-line of which the bob is at P_0; the meridian plane is the plane which passes through the vertical and is parallel to the axis of rotation of the earth; the east-and-west line is the normal to the meridian plane; the latitude is the complement ϕ of the angle which the vertical (to zenith) makes with the axis (to north); the longitude is the angle λ which the meridian plane makes with a fixed plane through the axis. See Pizzetti, Geodesia Teoretica, 1905, § 5.

On a level surface, that is, a surface at each point of which the normal is the vertical, a curve of constant latitude is a locus for which ϕ is constant, for the equator $\phi = 0$; a curve of constant longitude is a locus for which λ is constant; an east-and-west curve is a locus at each point of which the tangent is the east-and-west line. In general, a tangent to a curve of constant latitude is not an east-and-west line. With the foregoing understanding of the terms involved, Professor Roever shows:

I. The curves of constant latitude are the lines of constant intensity of illumination on a level surface, with respect to a source of light infinitely far away in the direction of the axis. In particular, the equator is the line of shade.

II. The curves of constant longitude are the loci of the brilliant points of the east-and-west curves on a level surface, with respect to infinitely distant pairs of points. That is, if a level surface of the earth, supposed metallic, were scratched along its east-and-west curves, an observer on the moon, or any other distant body in any general direction, would see as a curve of light some curve of constant longitude, the curve being the assemblage of the images of the sun, which may be in any general direction, in the reflecting scratches.

Definitions of brilliant points, and photographs of the curve of light formed by the images of a source of light in the scratches which a circular saw receives during the process of polishing, are given in a paper by the author in the *Transactions*, volume 9, number 3, pages 245–279. A brass sphere, properly scratched, was exhibited in order to show the truth of the second theorem for a special case.

3. In Professor Roever's second paper the first mechanism consists of a pair of overlapping wheels, about $8\frac{1}{2}$ inches in diameter, with radial spokes. These are mounted on parallel axles and rotate in planes which are as near together as possible. These wheels can be made to rotate in the same direction or in opposite directions with equal or different angular velocities. When the wheels rotate rapidly, an observer properly situated will see a well-defined system of curves.

The second mechanism is so designed that the effect is the same as that which would be produced by the first mechanism in the limiting case in which the radius of one of the wheels becomes infinite. The author has already shown for what fields of force the systems of curves exhibited by these mechanisms are the lines of force. See *Transactions of the Academy of Science of St. Louis*, volume 7, numbers 9 and 12, and BULLETIN, volume 12, page 425.

In his Researches in Chemistry and Physics, page 292, Faraday describes the phenomenon observed in viewing one carriage wheel obliquely through another, and also that of viewing a rapidly running carriage wheel through a palisade or railing. Beyond stating that the curves in the first case

resemble the lines described by iron filings between the poles of a magnet, Faraday does not identify the curves with any particular fields of force.

4. In a recent paper* Professor Roever derived an expression for the southerly deviation of falling bodies, under the assumption of a distribution of revolution, that is, that the potential function of the earth's gravitational field of force is of the form $f(r, z)$, where r is distance from the axis of rotation and z is that from a fixed plane perpendicular to the axis. He showed that for a potential function for which the standard spheroid is a level surface formula (3) below gives values nearly five times as great as the formula of Gauss,

$$2\omega^2 \sin 2\phi \frac{h^2}{6g}.$$

In the present paper he derives, under the assumption of a distribution not of revolution, the following expressions for the easterly deviation (E. D.) and the southerly deviation (S. D.):

$$(1) \quad \text{E. D.} = \frac{2}{3}\sqrt{2}\omega \cos\phi \frac{h^{\frac{3}{2}}}{\sqrt{W_z}} - \frac{5}{6}\frac{W_{yz}}{W_z}h^2,$$

$$(2) \quad \text{S. D.} = [2\omega^2 \sin 2\phi + 5W_{zz}]\frac{h^2}{6W_z} - \frac{\sqrt{2}}{15}\omega[9 \sin\phi W_{yz} + \cos\phi W_{xy}]\frac{h^{\frac{3}{2}}}{W_z^{\frac{1}{2}}},$$

where ω is the angular velocity of the earth's rotation, h is the height through which the body falls, ϕ is the latitude of the initial point P_0, W is the potential function of the relative statical field, that is, the field of force in which the plumb line is in equilibrium, x, y, z are distances measured to the north, east, and downward from P_0, and W_z, W_{zz}, W_{yz}, W_{xy} are the values of the derivatives of W, with respect to the indicated subscripts, at P_0. Hence

$$W_z = g, \quad W_{zz} = \frac{\partial g}{\partial x}, \quad W_{yz} = \frac{\partial g}{\partial y},$$

where g is the acceleration.

* *Transactions*, vol. 12, No. 3, pp. 335–353.

For a distribution of revolution $W_{yz} = 0$, $W_{xy} = 0$; hence the second terms disappear, being negligible in comparison with the first, that is,

$$\text{S. D.} = \left(2\omega^2 \sin 2\phi + 5\frac{\partial g}{\partial x}\right)\frac{h^2}{6g}.$$

The second derivatives of W which appear in (1) and (2) can be measured experimentally by a method due to Baron Eötvös.* In a certain survey made by Baron Eötvös† values of $\partial g/\partial x$ were found more than ten times as great as the value which corresponds to the potential function which yields the standard spheroid. It is thus seen that it is possible to account for southerly deviations which are nearly fifty times as great as those given by the formula of Gauss.

5. A conformal rational algebraic transformation between two planes may be established by an irreducible rational fraction $w = f(z)/g(z)$ of the complex variable z. Writing $f(z) = u + iv$, $g(z) = r + is$, and separating real from imaginary parts,

$$w = \frac{ru + sv}{r^2 + s^2} + i\frac{rv - su}{r^2 + s^2}.$$

Professor Emch's paper deals with the properties of the system of curves $u = 0$, $v = 0$, $r = 0$, $s = 0$, $ru + sv = 0$, $rv - su = 0$, $u^2 + v^2 = 0$, $r^2 + s^2 = 0$ and their relation to the complex domain. In conclusion some geometrical applications of particular cases are given.

6. This paper forms part of an account of the theory of the satellites of the solar system prepared by Professor Laves for the sixth volume of the Encyklopädie der mathematischen Wissenschaften. It calls attention to the very important recent improvements in our knowledge of this much neglected subject. Indications are not lacking that the theory of Laplace-Souillart must in the near future be replaced by a completely new theory, which abandons the method of variation of constants and takes the equator of Jupiter as the

* *Encyklopädie der mathematischen Wissenschaften*, Band VI, 1 B., Heft 2, § 23, p. 166.

† Verhandlungen der 15. allgemeinen Conferenz der internationalen Erdmessung 1906, 1, p. 337.

fundamental plane, and not Jupiter's orbital plane, which has so far been the adopted plane of reference. It stands to reason that Professor Griffin's paper in volume 9 of the *Transactions* will form the foundation for this new theory for the satellites I, II and III.

7. In a paper read at the April, 1910, meeting of the Section, Professor Curtiss proved the theorem that for any polynomial $f(x)$ there exist polynomials $\phi(x)$ such that the number of variations of sign in the product $f(x)\phi(x)$ is exactly equal to the number of positive roots of $f(x)$. Before this paper was published, an article by Meissner appeared in the *Mathematische Annalen*, discussing the case where $f(x)$ has no positive roots, and introducing certain geometrical methods. This suggested new phases of the problem which are discussed in the present communication. A second proof of the fundamental theorem is given, based on a well-known theorem due to Laguerre. The determination of all cartesian multipliers, $\phi(x)$, of given degree r, is made to depend upon properties of a configuration of linear $(r-1)$-spaces in space of r dimensions. Preliminary evaluations for the least degree possible for a cartesian multiplier are made, and various formulas are compared in a simple case.

8. The proofs generally given of several theorems of fundamental importance in solid geometry are based on intuitive relations assumed to exist between geometrical and physical objects, and are not admissible in a rigorous treatment of geometry. Among these are: dihedrals having equal plane angles are equal; symmetrical trihedrals are equal if isosceles, otherwise not; two prisms are equal if a trihedral of one is equal to a trihedral of the second, and if corresponding face angles of the equal trihedrals are corresponding angles of equal faces of the prisms.

Professor Whittemore shows that rigorous proofs of these and other theorems may be given on the basis of two simple assumptions and the following definition of equality of two geometrical configurations: Two configurations C and C' are equal if there is a one to one correspondence of all points A of space to all points A' of space such that (1) if A is on C, A' is on C', and conversely, if A' is on C', A is on C; (2) all relations of number and order of points A are identical with the relations of number and order of points A'.

9. In this paper Professor James, following the method of Seeliger, considers the effect of an assumed motion of the empirical trihedron of gravitational astronomy relative to the inertial on the outstanding differences between observation and theory in the motion of the four inner planets as derived by Newcomb. After correction for this motion, the effect of replacing the newtonian law of attraction by one consistent with the relativity theory is studied and the equations of motion of an infinitesimal planet about the sun

$$\frac{d^2x_i}{dt^2} = \frac{k^2M}{r^3}(-x_i + r\epsilon\xi + x_i\phi^2)\left(\frac{1}{\sqrt{1-\phi^2}}\right)^n \quad (i = 1, 2, 3),$$

set up by Professor de Sitter, are integrated and the exponent n so determined as to remove the excess in the motion of Mercury's perihelion. It is shown that the empirical terms in the motion of the four inner planets then become less than their mean errors.

10. If the mean anomaly M, which is proportional to the time, and the radius vector r of the planet be considered as rectangular coordinates, the (M, r) curve is a trochoid. This theorem, due to Newton, seems to have been almost universally forgotten. Professor Wilczynski points out that this theorem is important in the first place as giving an intuitive notion of the functional relation between the radius vector and the time, and in the second place as offering the possibility for the construction of a simple instrument by means of which r may be obtained without any calculation whatever. Thus, the solution of Kepler's equation would become altogether unnecessary if the instrument were constructed with the requisite degree of accuracy. In all cases, however, even a moderately accurate machine of this kind would be of great assistance by giving a close approximation to the value of the eccentric anomaly E, the final value then to be obtained by the application of the usual differential formulas.

11. If we denote by k any algebraic number field, and let α be any integer in k which is not the lth power of an integer in the same field, where l is a rational odd prime, then $\sqrt[l]{\alpha}$ determines a field K of degree l relative to k. Dr. Wahlin gives in this paper a complete determination of the decom-

position of the prime ideals of k into their prime factors in K. The paper is a generalization of the one presented by the author at the April meeting, 1911, as well as of the first part of the article by Westlund in the *Transactions*, 1910. The same problem for the case $l = 2$ is solved in an article by Hilbert in the *Mathematische Annalen*, volume 51.

12. This paper is a continuation of the work which Dr. Bill presented at the December, 1911, meeting of the Chicago Section. Having classified all the analytic curves of non-euclidean space and derived the Frenet-Serret differential equations for the so-called regular curves, he comes to a discussion of the natural equations of this latter set of curves.

13. Johannes de Muris, who flourished in the first half of the fourteenth century, wrote extensively on music as well as mathematics. His "Quadripartitum numerorum" has been referred to by many writers, but only a small portion of the second book dealing with arithmetic has been published. Professor Karpinski continues his studies in medieval algebra by a discussion of the contents of the third book of the Quadripartitum, which deals with algebra. This book is entitled "Novus algorismus de additis et diminutis." The study shows that de Muris drew not only from Al-Khowarizmi, but further that he took many problems almost word for word from Leonardo of Pisa. Inserted between the third and the fourth books is a so-called Semi liber which treats of the extraction of roots and again follows Leonardo. This study is based upon the Codex Pal. Vind. (Vienna), 4770, in which the third book is found on pages 229ver.—261rec.

14. A group is said to be divisible if it is the direct product of two or more groups none of which is identity. If it is not such a direct product it is said to be indivisible or prime. If a group is divisible it can be resolved into indivisible factor groups in essentially only one way. Professor Miller considers mainly various infinite systems of indivisible groups which are such that no two groups of the system have the same order. Among the theorems established are the following: A necessary and sufficient condition that a solvable group is a direct product of a Sylow subgroup and another subgroup is that its group of inner isomorphisms involves the corresponding

Sylow subgroup as a factor of a direct product, whenever it involves such a Sylow subgroup. A group of order p^m, p being any prime number, whose central is cyclic is always indivisible. A necessary and sufficient condition that the holomorph of a cyclic group is indivisible is that the order of this cyclic group is a power of a single prime number.

15. In this paper Mr. Denton presents the details of a projective differential theory of developables, which was indicated by Professor Wilczynski at the April, 1909, meeting of the Chicago Section.*

16. Professor Baker's paper has for its aim the establishment of an effective technique. It is proved that ordinary points on the discriminantal locus give rise to simple transpositions as elements of the monodromy group. The singularities, including the intersection of the various sheets in the case of reduction, give any other types of substitution which arise. By birational transformation of the resolvent equations three parameter quartics of each of the transitive groups are made available as material. In the case of non-symmetric groups, this can be so arranged that one parameter is removable by a rational process and is ineffective for monodromy. Examples are given of all cases of irreducible three-parameter quartics with assigned group and assigned invariant subgroup as monodromy group; also of the determination of the algebraic number whose adjunction reduces the group to the monodromy group.

17. Following the classification made by Moore in his fundamental paper † on transcendentally transcendental functions Professor Carmichael calls a transcendental function y of x algebraically or transcendentally transcendental according as it does or does not satisfy an algebraic differential equation. In the first part of the paper attention is confined to functions y of x expansible in the form

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots,$$

where $a_0, a_1, a_2, \cdots$ are rational fractions in their lowest terms.

* BULLETIN, vol. 15, p. 436.
† *Math. Ann.*, vol. 48, pp. 49–74.

Among the general results obtained the following may be mentioned: (1) If y is an integral of an algebraic differential equation, then there exists an integer τ such that, for increasing n, the denominator of a_n ultimately becomes and remains less than $n^{\tau n}$. (2) If y is an integral of an algebraic differential equation and if the denominator of a_n is written in the form $r_n s_n$, where s_n is prime to $n!$ and to the denominator of every $a_i, i < n$, then there exists an integer ϵ such that, for increasing n, s_n ultimately becomes and remains less than $n\epsilon$. (3) Let $b_0, b_1, b_2, \cdots$, be a set of rational fractions in their lowest terms, an infinite number of which are different from zero; let them have the property that a number N exists such that, for $n > N$, neither the numerator nor the denominator of b_n contains a prime factor greater than n. Let α and β be relatively prime positive integers. Then the series

$$y = b_0 + \frac{b_1}{(\alpha - \beta)^k} x + \cdots + \frac{b_n}{(\alpha - \beta)^{k_{1n}}(\alpha^2 - \beta^2)^{k_{2n}} \cdots (\alpha^n - \beta^n)^{k_{nn}}} x^n + \cdots$$

if it converges, defines a function which is transcendentally transcendental. Here k_{ij} is an integer and $k_{nn} \geqq 1$ for an infinite number of values of n for which $b_n \neq 0$. (4) Particular examples of functions coming under this theorem are those which are analytic at zero and which satisfy the following functional equations, where q is rational and $|q| \neq 1, 0$:

$$g(qx) = (1+x)g(x), \quad g(qx) - g(x) = \frac{x}{1-x}, \quad e^x - 1, \quad \sin x, \quad \text{etc.}$$

In the second part of the paper it is shown that every analytic function integral of the equation

$$g(qx) = (1 + x)g(x), \quad |q| \neq 1, 0,$$

is transcendentally transcendental. Finally the following important result is obtained: Two essentially different classes of transcendentally transcendental functions are defined by linear difference equations with rational coefficients and by linear q-difference equations with rational coefficients.

18. Guichard, Appell, and Hurwitz have each proved by indirect means the important theorem that if $G(x)$ is any

entire function then there exists another entire function $u(x)$ verifying the relation

$$u(x+1) - u(x) = G(x).$$

In the first part of his second paper Professor Carmichael proves this theorem by direct means; that is to say, a power series is assumed for $u(x)$ and substitution is made in the above equation. There results an infinite system of equations for determining the coefficients of this power series. This system is solved by what is believed to be a novel method of importance, and in such way that the resulting function $u(x)$ is entire.

The remainder of the paper is devoted to certain generalizations of the above result.

19. Professor Glover derives a formula for the valuation of a very general type of security. The security is redeemed in r equal instalments at intervals of t years, the first redemption being made after f years. The annual rate of dividend is g payable in m instalments and the security is purchased to realize the investor a nominal rate of interest with frequency of conversion m. The value is expressed in terms of a single function, namely, the present values of immediate annuities certain.

20. Dr. Dodd shows that the conditions of continuity, differentiability, and convexity imposed by Pringsheim * in defining an ordinary arc and its function $y = f(x)$ are not coextensive with the conditions used by Du Bois-Reymond,† who, in place of convexity, requires that there should not be an infinite number of maxima with respect to any straight line. But Pringsheim's conditions necessitate the continuity and univariance of the first derivative except at the end points of the interval, where being univariant it may become infinite. Pringsheim's ordinary arc is Osgood's ‡ regular arc defined by

$$x = \varphi(t), \quad y = \psi(t), \qquad (t_0 \leqq t \leqq t'),$$

* Encyklopädie, II, A. 1, p. 22.
† *Crelle*, vol. 79, p. 32.
‡ Lehrbuch der Funktionentheorie, p. 43.

where φ and ψ are continuous and have continuous first derivatives which do not simultaneously vanish in (t_0, t'), with the added condition that ψ'/φ' be univariant in (t_0, t').

The definition of an ordinary arc is then restated to make it invariant to rotation, a generalization for three dimensions is given, and finally straight lines, ordinary plane arcs, and ordinary skew arcs are distinguished by means of the Wronskians of the first derivatives of the defining functions, and the minors of the Wronskians.

21. Professor Cajori points out that, at the beginning of the nineteenth century and for three hundred years previous, Spanish-American manuscripts often contain, along with the Hindu-Arabic numerals, symbols taking the form of a U or V or O, or sometimes an old form of the Greek letter sigma, to designate "thousands." Thus, 7U291 = 7291.

22. In the "Introduction to general analysis" E. H. Moore has formulated a theory of developments of a general range P. A development of P is defined to be a denumerably infinite sequence of stages, each stage consisting of a finite system of subclasses of P. In terms of a development Moore and A. D. Pitcher have secured a body of theorems relating to classes of functions on a general range. Mr. Chittenden considers developments whose stages are denumerably infinite systems of subclasses and obtains theorems which include those of Moore and Pitcher and cover additional cases.

24. The projective differential geometry of ruled surfaces in ordinary space has been developed by Professor Wilczynski by means of a system of two linear homogeneous differential equations of the second order. In the present paper Dr. Stouffer develops by similar methods the projective differential geometry of three-spreads generated by ∞^1 planes in five-dimensional space by making use of a system of three linear homogeneous differential equations of the second order. The three-spreads for which any two consecutive generating planes intersect are excluded from the theory.

25. Consider sequences α: $a_1, a_2, a_3 \cdots$ of positive terms and denote by $\mathfrak{C}_u$, where u is a non-vanishing real number, the class of all sequences α whose u power sequences α^u of positive terms

have convergent sums $a_1^u + a_2^u + a_3^u + \cdots$, and by $\mathfrak{D}_u$ the class of all sequences α whose u power sequences α^u have divergent sums. Then, d, e, f being three positive numbers such that $(d + e)f = d + e$, we have the multiplication theorems

$$\mathfrak{C}_d\mathfrak{C}_e = \mathfrak{C}_f, \quad \mathfrak{C}_e\mathfrak{D}_{-f} = \mathfrak{D}_{-d}.$$

These theorems are convenient normalizations, slightly generalized, of theorems due to Pringsheim* and Landau,† which especially for $(d, e, f) = (2, 2, 1)$ play a rôle in the theory of functions of infinitely many variables.

Professor Moore determines a closed multiplication table for a set of classes of sequences of positive terms whose definitions involve, besides the notions u power and convergence, the notions finiteness and exponent of convergence.

26. Let $f(x)$ be continuous and $|f(x)| \leqq M$ for $0 \leqq x \geqq 2\pi$, and suppose that the Fourier coefficients of $f(x)$ satisfy the inequalities

$$|a_n| \leqq A/n, \quad |b_n| \leqq B/n \quad (n = 1, 2, \cdots).$$

Let

$$s_n(x) = a_0 + a_1 \cos x + b_1 \sin x + \cdots + a_n \cos nx + b_n \sin nx$$

be a partial sum of the Fourier series for $f(x)$; Fejér has shown that

$$|s_n(x)| \leqq M + \frac{1}{n+1}\left(A\sum_{\nu=1}^{n}|\cos \nu x| + B\sum_{\nu=1}^{n}|\sin \nu x|\right),$$

whence

$$|s_n(x)| < M + A + B.$$

In Dr. Gronwall's paper, this inequality is replaced by the following closer ones:

$$|s_n(x)| < M + \sqrt{A^2 + B^2}$$

for any values of A and B, and

$$|s_n(x)| < M + \tfrac{1}{2}A + \frac{\sqrt{3}}{2}B$$

for $B \geqq \sqrt{3}A$.

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* *Münchener Berichte*, vol. 32 (1902), p. 177.

† *Göttinger Nachrichten*, 1907, pp. 25–27.

Making

$$U_n(x) = \frac{1}{n+1}(|\cos x| + |\cos 2x| + \cdots + |\cos nx|),$$

$$V_n(x) = \frac{1}{n+1}(|\sin x| + |\sin 2x| + \cdots + |\sin nx|),$$

it is further shown that $\lim_{n=\infty} U_n(x)$ and $\lim_{n=\infty} V_n(x)$ are pointwise discontinuous functions of x, having the following values: For $x = 0$,

$$\lim_{n=\infty} U_n(x) = 1, \quad \lim_{n=\infty} V_n(x) = 0;$$

for $x = \frac{p}{q}\pi$, where $p\ (\neq 0)$ and q are integers and relative primes,

$$\lim_{n=\infty} U_n(x) = \frac{1}{2q}\left(\cot\frac{\pi}{4q} + (-1)^{q+1}\tan\frac{\pi}{4q}\right),$$

$$\lim_{n=\infty} V_n(x) = \frac{1}{q}\cot\frac{\pi}{2q};$$

for $x : \pi$ irrational,

$$\lim_{n=\infty} U_n(x) = \lim_{n=\infty} V_n(x) = \frac{2}{\pi}.$$

It is finally shown that

$$\tfrac{1}{2} = \text{min.}\lim_{n=\infty} U_n(x) = \lim_{n=\infty}\text{min. } U_n(x) < \text{max.}\lim_{n=\infty} U_n(x) =$$

$$= \lim_{n=\infty}\text{max. } U_n(x) = 1,$$

$$0 = \text{min.}\lim_{n=\infty} V_n(x) = \lim_{n=\infty}\text{min. } V_n(x) < \text{max.}\lim_{n=\infty} V_n(x) =$$

$$= \frac{2}{\pi} < \lim_{n=\infty}\text{max. } V_n(x) = 0.724\cdots,$$

$$\text{max. } V_n(x) < \text{max. } V_{n+1}(x),$$

so that $V_n(x)$ presents an analogue to Gibbs' phenomenon, while there is none in the case of $U_n(x)$.

27. Let

$$s_\alpha(x) = \sum_{d/x} d^\alpha$$

be the sum of the αth powers of all divisors of the integer x. Dr. Gronwall shows that

$$\limsup_{x=\infty} \frac{s_\alpha(x)}{x^\alpha} = \xi(\alpha) \quad \text{for } \alpha > 1,$$

$$\limsup_{x=\infty} \frac{s_\alpha(x)}{x \log\log x} = e^C \quad \text{for } \alpha = 1,$$

$$\limsup_{x=\infty} \frac{\log s_\alpha(x)/x^\alpha}{(\log x)^{1-\alpha}/\log\log x} = \frac{1}{1-\alpha} \quad \text{for } 0 < \alpha < 1,$$

where $\xi(\alpha)$ is the Riemann zeta function and C the Eulerian constant.

28. In this paper Professor Moulton proves the existence of orbits of ejection which are also orbits of collision with the same finite body. These orbits are the limits of two classes of periodic orbits.

In his Stockholm lectures Painlevé conjectured that an orbit satisfying a certain analytic condition would be an orbit of ejection or collision. In this paper it is proved that any point and an arbitrary positive velocity may be selected, and that an orbit of collision and one of ejection exist which pass through the point with the given velocity. The direction of motion is an analytic function of the coordinates of the point and the velocity.

29. The chief theorem in Dr. Hildebrandt's paper is the following: Suppose $m > 0$,

$$\sum_p |x_{np}|^m$$

convergent for every value of n, and

$$\lim_n x_{np} = x_p$$

for every value of p. Then the necessary and sufficient condition that

$$\sum_p |x_p|^m$$

be convergent and

$$\lim_n \sum_p | x_{np} |^m = \sum_p | x_p |^m$$

is that

$$\lim_n \sum_p | x_{np} - x_p |^m = 0.$$

In the proof of this theorem there arises another necessary and sufficient condition concerning itself with the limit of the sum of a sequence of series of positive terms.

H. E. SLAUGHT,
Secretary of the Section.

THE TWENTY-FIRST REGULAR MEETING OF THE SAN FRANCISCO SECTION.

THE twenty-first regular meeting of the San Francisco Section of the Society was held at Stanford University, on Saturday, April 6, 1912. About fifteen persons were present, including the following members of the Society:

Professor R. E. Allardice, Mr. B. A. Bernstein, Dr. Thomas Buck, Professor H. F. Blichfeldt, Professor G. C. Edwards, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Professor H. C. Moreno, Professor E. W. Ponzer.

Morning and afternoon sessions were held, Professor Hoskins, chairman of the Section, presiding.

The following papers were presented at this meeting:

(1) Mr. B. A. BERNSTEIN: "On the relation between spaces in n-dimensional space and their concrete representation for the space of four dimensions."

(2) Dr. THOMAS BUCK: "Some periodic orbits of three finite bodies."

(3) Dr. S. LEFSCHETZ: "On cubic surfaces and their nodes."

(4) Professor H. F. BLICHFELDT: "On the order of linear homogeneous groups. Fifth paper."

(5) Mr. B. A. BERNSTEIN: "On an algebra of probability" (preliminary communication).

In the absence of the author the paper by Dr. Lefschetz was read by title. Abstracts of the papers follow.

1. In this paper Mr. Bernstein derives synthetically theorems concerning the fundamental relations between spaces in a hyperspace, including general formulas relating to the dimensionality of any space in any other space as element. He then shows how those relations can be represented concretely for four-dimensional space.

2. The orbits considered by Dr. Buck are obtained by the method of continuation as to a parameter which has been developed by Professor Moulton in connection with similar problems. The results are given as power series in the parameter which are known to converge for all values of the parameter sufficiently small. When the parameter is put equal to zero the solutions found reduce to the circular Lagrangian equilateral triangle solution of the problem of three bodies. For the other admissible values of the parameter the motion may be described as an oscillation about the vertices of this triangle.

3. In this paper Dr. Lefschetz gives a treatment of the cubic surface S^3 derived from a generation given by R. Sturm. The surface is obtained as the locus of the variable twisted cubic common to two quadrics which pass respectively through two other twisted cubics R_1^3 and R_2^3, having a point O in common, and which also pass through a common variable line through O. It is shown by the principle of correspondence that both curves have 6 common bisecants, not going through O, and which are therefore on S^3. The existence of the other 21 lines on S^3 and their properties are readily established. The modifications necessary in order to obtain the 21 non-ruled cubic surfaces are given except for the surface U_8 (Schlafli's classification—Salmon, Geometry of Three Dimensions, page 395), for which it is here proved that it contains no twisted cubic, being the only S^3 having this property. For the three surfaces with a node (U_6, U_7, U_8) a second construction is given, based on a modification of the construction of an S^3 by projective pencils of quadrics and planes (Steiner). In addition to their being new, the interest in these constructions consists first, in the geometric classification of cubic surfaces, according to the mutual position of R_1^3 and R_2^3 in space, and second, in making it possible to show the properties of each of the surfaces without considering them as limiting cases. This is illustrated for the case of the surface with a binode.

4. On the basis of a principle discovered by Bieberbach, Frobenius proved the following theorem (*Berliner Sitzungsberichte*, 1911, page 373): Let a finite group G of linear homogeneous substitutions contain a substitution S whose "multipliers" (i. e., the roots of its "characteristic equation," necessarily roots of unity) $\alpha_1, \alpha_2, \cdots, \alpha_n$ possess the property arc $(\alpha_i/\alpha_j) < 60°$; then all the substitutions of G of this character are mutually commutative. It follows that G is not primitive. Professor Blichfeldt's paper, based on the same principle, contains a number of theorems of which the following are noted here:

(1) If arc $(\alpha_i/\alpha_j) \leqq 72°$, then G is not primitive.

(2) No abelian subgroups of a primitive collineation group in n variables can be of order $\geqq 5^{n-1}$.

This maximum order, of importance in fixing a limit to the order of collineation groups, is presented in much lower (and correspondingly much more complex) form for large values of n. A more general theorem is given, of which the following is a special case:

(3) If the characteristic equation of a substitution of prime order p has only three distinct roots, then $p \leqq 13$, or the group is not primitive.

5. The theory of probability has underlying it both numerical and logical assumptions. Mr. Bernstein, in his second paper, presents a set of postulates for a numerico-logical theory of probability and establishes on this foundation a general method of solving the problem: Given $p_1, p_2, \cdots, p_n$, the probabilities of the events $E_1, E_2, \cdots, E_n$; to find the probability x of the event $E = f(E_1, E_2, \cdots, E_n)$.

T. M. PUTNAM,
Secretary of the Section.

IMPLICIT FUNCTIONS DEFINED BY EQUATIONS WITH VANISHING JACOBIAN.

BY MR. G. R. CLEMENTS.

(Read before the American Mathematical Society, April 28, 1911.)

LET

$$x_i = f_i(y_1, \cdots, y_n) \qquad (i = 1, 2, \cdots, n)$$

be n functions, each analytic in the neighborhood of the point $(y) = (0)$. The case in which the Jacobian

$$J = \sum \pm \frac{\partial f_1}{\partial y_1} \frac{\partial f_2}{\partial y_2} \cdots \frac{\partial f_n}{\partial y_n}$$

does not vanish in the point $(y) = (0)$, and the case in which J vanishes identically in the neighborhood of this point, are familiar. I wish to state here some theorems which I have obtained for the intermediate case that J vanishes in the point $(y) = (0)$, but does not vanish identically in the n variables y. For a number of the theorems n is restricted to the value 2.

If $x_1, \cdots, x_n$ denote n independent complex variables, any region which constitutes at least the totality of all points $(x_1, x_2, \cdots, x_n)$ for which

$$| x_1 - a_1 | < \epsilon, | x_2 - a_2 | < \epsilon, \cdots, | x_n - a_n | < \epsilon$$

for some $\epsilon > 0$, is called a *complete* neighborhood of the point $(a_1, a_2, \cdots, a_n)$.

Any region which satisfies the conditions that

(1) for every $\epsilon > 0$, it contains points $(x_1, x_2, \cdots, x_n)$ for which

$$| x_1 - a_1 | < \epsilon, | x_2 - a_2 | < \epsilon, \cdots, | x_n - a_n | < \epsilon;$$

(2) for no $\epsilon > 0$, *every* point $(x_1, x_2, \cdots, x_n)$ for which

$$| x_1 - a_1 | < \epsilon, | x_2 - a_2 | < \epsilon, \cdots, | x_n - a_n | < \epsilon$$

is a point of the region

is called a *partial* neighborhood of the point $(a_1, a_2, \cdots, a_n)$.

For purposes of simplicity, the neighborhood of the point $(0, 0, \cdots, 0)$ is considered throughout this paper. In what

follows, the unqualified word *neighborhood* is used to mean *complete neighborhood.*

Consider the transformation

$$T: \qquad x = f(u, v), \quad y = \varphi(u, v),$$

(*a*) $f(u, v)$ and $\varphi(u, v)$ denoting functions of the complex variables u and v, single-valued and analytic throughout a neighborhood R of $u = 0, v = 0$, and satisfying the conditions

(*b*) $$f(0, 0) = 0, \quad \varphi(0, 0) = 0,$$

(*c*) $$J(u, v) \equiv \begin{vmatrix} f_u & f_v \\ \varphi_u & \varphi_v \end{vmatrix} \not\equiv 0, \quad J(0, 0) = 0.$$

To any point (u, v) of R, there corresponds, under T, one and but one point (x, y) of the (x, y) space. The totality of points (x, y) which thus correspond to points (u, v) of R, constitute a finite region $\bar{R}$ of the (x, y) space. In general, more than one point (u, v) will yield the same pair of values for x and y. We shall count a point (x, y) but once, thus regarding it as completely characterized by its coordinates, and seek an inverse transformation

$$u = \bar{f}(x, y), \quad v = \bar{\varphi}(x, y)$$

which will put in evidence all those points (u, v) of R which correspond under T to any point with coordinates (x, y) of $\bar{R}$. Thus the functions $\bar{f}(x, y)$, $\bar{\varphi}(x, y)$ are to be defined only in the points of $\bar{R}$, and there uniquely or as multiple-valued functions.

THEOREM I. *The transformation T can never have an inverse*

$$u = \bar{f}(x, y), \quad v = \bar{\varphi}(x, y)$$

such that $\bar{f}(x, y)$ and $\bar{\varphi}(x, y)$ are both analytic throughout a complete neighborhood of $x = 0, y = 0$.

The theorem holds for any value of n.

THEOREM II. *If in the transformation T, $f(u, v)$ and $\varphi(u, v)$ have no common factor in the point $(0, 0)$, then there exists an inverse, defined throughout a complete neighborhood of $x = 0$, $y = 0$, everywhere continuous, finitely multiple-valued, analytic except along a complex one-dimensional locus, and having the value $u = 0$, $v = 0$, when $x = 0, y = 0$.*

If instead of the transformation T, we consider the transformation T', whose definition differs from that of T in that it omits the condition (c), we deduce

THEOREM III. *If to each point (x, y) of the region $\bar{R}$ there corresponds from T' a single point (u, v) of a region R, then u and v are analytic in x and y in the point $x = 0, y = 0$.*

It follows from Theorem I and from Theorem III that there can not exist a transformation T for which the inverse is single-valued throughout any neighborhood $\bar{R}$ of the point $x = 0, y = 0$.

The proof for Theorem II and for Theorem III, depends on the Weierstrassian implicit function theorem.* Theorem IV is a generalization of this.

THEOREM IV. *The system of equations*

$$f_1(x_1, \cdots, x_n; y_1, \cdots, y_p) = 0,$$

$$f_2(x_1, \cdots, x_n; y_1, \cdots, y_p) = 0,$$

$$\cdots\cdots\cdots\cdots$$

$$f_p(x_1, \cdots, x_n; y_1, \cdots, y_p) = 0,$$

where

(1) *f_i is an analytic function of its arguments throughout a neighborhood R of $(x) = 0, (y) = 0$;*

(2) $f_i(0, \cdots, 0; 0, \cdots, 0) = 0 \quad (i = 1, 2, \cdots, p)$;

(3) $$J_1 \equiv \frac{D(f_1, f_2, \cdots, f_p)}{D(y_1, y_2, \cdots, y_p)} = 0 \text{ when } (x) = 0, (y) = 0,$$

$$\cdots\cdots\cdots\cdots$$

$$J_{k-1} \equiv \frac{D(J_{k-2}, f_2, \cdots, f_p)}{D(y_1, y_2, \cdots, y_p)} = 0 \text{ when } (x) = 0, (y) = 0,$$

$$J_k \equiv \frac{D(J_{k-1}, f_2, \cdots, f_p)}{D(y_1, y_2, \cdots, y_p)} \neq 0 \text{ when } (x) = 0, (y) = 0,$$

defines $y_1, y_2, \cdots, y_p$ as functions of $x_1, x_2, \cdots, x_n$, continuous throughout the neighborhood of $(x) = 0$. With a suitable counting of multiplicities everywhere present, this solution is k-fold; it is analytic with the possible exception of complex $(n-1)$-dimensional loci where determinations of the root, in general distinct, become coincident. When $(x) = 0, (y) = 0$.

*Weierstrass, Abhandlungen aus der Funktionenlehre, p. 105; Bliss, BULLETIN, April, 1910, p. 356; Macmillan, ibid., December, 1910, p. 116.

THEOREM V. *The transformation*

$$x_1 = f_1(y_1, \cdots, y_n), \quad x_2 = f_2(y_1, \cdots, y_n), \cdots, x_n = f_n(y_1, \cdots, y_n),$$

where

(1) f_i *is a single-valued and analytic function of* $y_1, \cdots, y_n$ *in the point* $(0, \cdots, 0)$ *and vanishes there* $(i = 1, 2, \cdots, n)$;

$$(2) \qquad J_1(0, \cdots, 0) = \frac{D(f_1, \cdots, f_n)}{D(y_1, \cdots, y_n)}\Bigg|_{(y)=0} = 0,$$

$$\cdots \cdots \cdots \cdots \cdots \cdots$$

$$J_{k-1}(0, \cdots, 0) = \frac{D(J_{k-2}, f_2, \cdots, f_n)}{D(y_1, y_2, \cdots, y_n)}\Bigg|_{(y)=0} = 0,$$

$$J_k(0, \cdots, 0) = \frac{D(J_{k-1}, f_2, \cdots, f_n)}{D(y_1, y_2, \cdots, y_n)}\Bigg|_{(y)=0} \neq 0,$$

has a k-valued continuous inverse, defined throughout a complete neighborhood of $(x) = 0$. *This inverse is analytic, with k distinct determinations, except along a complex* $(n-1)$*-dimensional locus, where it is continuous and less than k-valued. When* $(x) = 0$, $(y) = 0$.

The remaining theorems have to do with a detailed study of the transformation T. Theorems VI, VII, and VIII yield a complete discussion for the case in which at least one element of the determinant J, say f_u, is not zero at the origin.

THEOREM VI. *If the transformation T satisfies the further conditions*

(1) $f_u(0, 0) \neq 0$,

(2) $f(u, v)$ *not a factor of* $\varphi(u, v)$ *in the point* $u = 0$, $v = 0$, *then, for the sequence of functions*

$$J_1(u, v) \equiv \frac{D(f, \varphi)}{D(u, v)}, \cdots, J_{i+1}(u, v) \equiv \frac{D(f, J_i)}{D(u, v)}$$

there exists an integer $k \geqq 2$ *such that*

$$J_1(0, 0) = \cdots = J_{k-1}(0, 0) = 0, \quad J_k(0, 0) \neq 0,$$

and for the transformation there exists a k-valued continuous inverse, defined throughout a complete neighborhood of $x = 0$, $y = 0$. *This inverse is analytic, with k distinct determinations, except along a complex one-dimensional locus, where it is continuous and less than k-valued. Finally* $u = 0$, $v = 0$ *when* $x = 0$, $y = 0$.

THEOREM VII. *If the transformation T is of the form*

$$x = f(u, v), \quad y = [f(u, v)]^n g(u, v),$$

where

$$f_u(0, 0) \neq 0, \quad g(0, 0) = 0,$$

and where $f(u, v)$ is not a factor of $g(u, v)$ in the point $u = 0$, $v = 0$, then to the point $x = 0$, $y = 0$ corresponds the locus $f(u, v) = 0$, and to any other point (x, y)

(1) if $f_u g_v - f_v g_u \neq 0$ when $u = 0$, $v = 0$, there corresponds one and only one point (u, v), the relationship being defined by functions of x and y, single-valued and analytic in this point;

(2) if $f_u g_v - f_v g_u = 0$ when $u = 0$, $v = 0$, there correspond in general m points, $m \geqq 2$, the relationship being defined by m-valued continuous functions of x and y.

THEOREM VIII. *A transformation T for which $f(u, v)$ is a factor of $\varphi(u, v)$, can be replaced by a transformation*

$$u_1 = f(u, v), \quad v_1 = h(u, v),$$

where $h(u, v)$ is a single-valued and analytic function of u and v in the point $(0, 0)$ and vanishes there, and where $f(u, v)$ is not a factor of $h(u, v)$ in this point, followed by a finite number m of transformations of the form

$$x = u, \quad y = uv,$$

combined with not more than m translations of the form

$$x = u, \quad y = v + c.$$

THEOREM IX. *A transformation T which has the form*

$$x = f(u) = a_n u^n + a_{n+1} u^{n+1} + \cdots \quad (a_n \neq 0),$$
$$y = \varphi(u, v),$$

can be replaced by two transformations, $T = b \cdot a$, where

$$a: \quad u_1 = u \sqrt[n]{a_n + a_{n+1} u + \cdots}, \quad v_1 = \varphi(u, v),$$
$$b: \quad x = u_1^n, \quad y = v_1.$$

The inverse of (a) is defined by analytic functions, or can be determined from Theorems VI, VII *and* VIII.

THEOREM* X. *If $k = 2$ in Theorem* VI, *the transformation*

* Cf. L. S. Dederick, Harvard doctoral thesis (1909), "Certain singularities of transformations of two real variables," p. 124.

T can be replaced by three transformations which are one-to-one and analytic both ways, combined with one transformation of the form

$$x = u, \quad y = v^2.$$

THEOREM XI. *If the transformation T has the form*

$$x = f(u, v) \equiv c_{20}u^2 + c_{11}uv + c_{02}v^2 + c_{30}u^3 + \cdots,$$
$$y = \varphi(u, v) \equiv d_{20}u^2 + d_{11}uv + d_{02}v^2 + d_{30}u^3 + \cdots,$$

where the terms quadratic in u and v are not identically zero for either f or φ, and where these quadratic terms have no common factor, then there exists a four-valued continuous inverse, defined throughout the complete neighborhood of $x = 0, y = 0$. This inverse is analytic, with four distinct determinations, except along a complex one-dimensional locus, where it is continuous and less than four-valued. Finally, $u = 0, v = 0$ when $x = 0, y = 0$.

MADISON, WIS.

DARWIN'S SCIENTIFIC PAPERS.

Scientific Papers. By SIR GEORGE HOWARD DARWIN, K.C.B., F.R.S., Plumian Professor in the University of Cambridge. Vol. III, Figures of Equilibrium of Rotating Liquid and Geophysical Investigations, xv + 527 pp.; Vol. IV, Periodic Orbits and Miscellaneous Papers, xviii + 592 pp. Cambridge University Press, 1910, 1911. Royal 8vo.

THE first two volumes of Sir George Darwin's researches have already been reviewed in the columns of the BULLETIN.* They contained papers on the practical and theoretical tidal problems which the oceans present and his earlier attacks on the past history of the earth-moon system. The third and fourth volumes contain his investigations on the relations of fluid masses in rotation about an axis under gravitational forces, on the periodic orbits which a particle can describe when attracted by two bodies of finite masses moving in circular orbits about one another, and a number of papers on other matters.

Of the forms which a single mass of fluid can take when revolving without relative motion about an axis under its own gravitational attraction only, two have long been known. Maclaurin's ellipsoid is one of revolution about the axis of rotation and its eccentricity will have a value which depends

* Vol. 16, pp. 73–78.

on the rate of angular rotation. Jacobi's ellipsoids have three unequal axes. These forms are of little interest unless the motion is stable. As the eccentricity of the Maclaurin ellipsoid increases, the series reaches a point where stability changes to instability. At this place the Jacobian series changes from having three unequal axes to an ellipsoid having the two axes in the equatorial plane equal, that is, to the Maclaurin ellipsoid and, what is more important, from stability to instability.

If the stable portion of the Jacobian series be now followed from this point, there comes a place where the motion changes from stability to instability and at this place it crosses a third series, known as the pear-shaped figures of equilibrium which there degrade into a Jacobian ellipsoid. These pear-shaped figures were first known through a memoir by M. Poincaré which appeared in 1885.

Is it possible to carry the investigation further along the same lines so as to discover figures of relative equilibrium in which, as a certain constant alters, the fluid mass shall separate into two parts, the motion remaining stable? From the point of view of the development of the earth-moon and similar systems, this is the principal question.

While Poincaré was carrying the development to the stage mentioned above, Darwin had already attacked the problem from the other end. In the paper "On figures of equilibrium of rotating masses of fluid" he considers two equal masses revolving near to one another in relative equilibrium and examines the forms they will assume as they approach towards contact. By the use of spherical harmonic analysis he carries the solution to the stage where they are not only in contact but actually overlap. The last is of course a physical impossibility but it leads directly to the possibility of the existence of a figure formed of two masses joined by a narrow neck. The stability question is not, however, settled at this time. Several cases where the two masses are unequal are also considered.

In the attempt to use spherical harmonic analysis, the numerical applications become doubtful when the deviation from a sphere is too great. Darwin therefore undertakes a laborious and detailed investigation into the properties of ellipsoidal harmonics, so that a start may be made with an ellipsoid instead of with a sphere, and that the deviations from

the ellipsoidal form may be computed. More than a quarter of Volume III is devoted to this subject, but the space is not extravagant in view of the fact that the papers which follow mainly owe their success to the completeness with which he has developed the theory of these complicated functions and to the methods he has adopted in order that numerical results might be deduced from them.

The first step is a computation to find the critical Jacobian ellipsoid where this series crosses the pear-shaped series. It is found that the axes are nearly in the ratio 8 : 10 : 23, the actual values being given to five significant figures. The ellipsoidal harmonic analysis is then applied in order to find out in what way the pear-shaped figure develops from this critical case. It is shown that the principal changes take place at first in the longest axis. One end of this is lengthened and narrowed, the other shortened and blunted. In the following paper the approximation is carried to a higher order with the the result that the deviation from the ellipsoid is small everywhere except at the narrowed end and it there takes more nearly the form of a protuberance. Sir George Darwin conjectures that the further stage in the development would be the appearance of a furrow at this end, suggesting an ultimate detachment of a small portion of the mass. The general resemblance to a pear is much less striking than in M. Poincaré's conjectural figure drawn without a reduction to numbers.

The chief object of this second paper is, however, the determination of the stability of the series. Sir George Darwin comes to the conclusion that they form a stable series up to some point not determined. He is nevertheless careful to state M. Liapounoff's dissent from this conclusion in spite of the fact that a reexamination of his own work and a repetition of his calculations has not altered the coefficient by which the stability is determined.

In the last paper of volume III "On the figure and stability of a liquid satellite," the problems with which the series of memoirs originally started, as well as other kindred problems, are finally solved. The author opens with a tribute to Edouard Roche whose paper on fluid masses was published in 1847–50 at Montpellier but was neglected for many years. Roche's problem was the determination of the figure and stability of a liquid satellite revolving without relative motion in a circular orbit about a planet which is spherical and rigid.

Sir George Darwin replaces the rigid planet by a mass of fluid which can take the form appropriate for relative equilibrium, but he remarks that the analysis for both is practically the same up to the point where the stability has to be considered. He considers cases of different as well as of equal masses. In both problems the liquid assumes approximately ellipsoidal forms, so that the deformations may be conveniently determined by ellipsoidal harmonic analysis. At the limit of stability of the Roche figure, it is shown that the surfaces of the two bodies are separated by an interval which is not zero. When both the masses are liquid, the series of figures can be continued until they overlap, so that the dumbbell-shaped figure again appears as a possibility. Unfortunately, these figures before junction become very unstable and the author considers that it is inconceivable that a junction by a narrow neck of fluid should render them stable.

Thus the gap between two detached liquid masses and a single mass is not yet filled by stable figures of equilibrium. That one may not draw final conclusions concerning the genesis of a two-body system from this result is sufficiently certain. There are possibilities pointed out by the author—other figures of equilibrium, changes produced by heterogeneity of density, turbulence of motion during the fission,—not yet investigated, any one of which may furnish the desired solution of the problem.

The short paper entitled "The approximate determination of the form of Maclaurin's ellipsoid," published in the *Transactions* of the Society in 1903, is to be recommended as a beginning in a study of the methods employed to reach these conclusions. This problem, which of course admits of an exact solution, is treated by the methods employed in the more complicated cases. As the analysis is comparatively simple and is only carried to the degree of approximation necessary for clearness of exposition, it is easy to follow the several steps and to obtain a comprehensive view of the method.

One further paper on cosmogony is placed in Volume IV. Of this, Sir George Darwin says "I find it very difficult to estimate its value. If it is to be judged by the amount of comment to which a paper gives rise, it is a failure; for it has received but little notice." Perhaps an estimate fairer to its author would be that it was published (1889) somewhat ahead of the time when the need for such an investigation

became pressing. Briefly, it consists of an attempt to show that under certain conditions, fully defined, the nebular hypothesis of Laplace will lead to developments in the early history of a planetary system somewhat resembling those produced by the accretion of meteoric matter. While the former of these hypotheses has received a degree of attention commensurate with the time it has been before the scientific public, the latter is comparatively young and has yet to undergo extensive criticism and development before it may take a place beside its older rival. It is not hard to believe that before long this paper may form a foundation for the comparison of the two hypotheses and that the methods used therein may serve as a basis for a reconciliation or a distinction which may lead to the general acceptance of one or the other.

When Dr. G. W. Hill published in 1877 his now famous papers on the motion of the moon, probably neither he nor his early readers imagined that they would open up a new field in the study of problems concerning the motions of particles and bodies. He showed that it was possible, under certain restrictions, to obtain a closed orbit relative to certain moving axes and he gave a series of figures of such orbits for different values of the lunar month. M. Poincaré, taking up the idea, developed it in his essay of 1891 on the problem of three bodies, under the name of the periodic solution. Sir George Darwin, however, had evidently recognized their value before this time, for he made a recommendation to the reviewer to undertake a study of Hill's papers as a beginning likely to lead to results. He himself, however, published nothing on the subject until 1897 when the long paper entitled "Periodic orbits" appeared in the *Acta Mathematica*. Dr. Hill had taken the disturbing body (the sun) of infinite mass moving uniformly round the earth at an infinite distance, and had considered the periodic motion of a satellite of the earth; he finds a portion of a single family of orbits, the moon belonging to one member of the family. Sir George Darwin took the sun to be moving uniformly in a circular orbit at a finite distance and to be of mass equal to ten times that of the earth or planet, and he studies not only the satellite orbits of the third body but also the planetary orbits which are periodic. The main difficulty arises from the adoption of so small a mass of the sun relative to the planet. The ordinary methods of development in series become inapplicable

owing either to lack of or to extreme slowness of convergence. He therefore resorted to mechanical quadratures. Starting at an arbitrary place on the sun's radius vector with a given velocity, the orbit is traced piece by piece until it crosses the axis on which it started. If it proves to be reentrant, the periodic orbit is obtained; if not, the starting point is changed and a new orbit computed. After several attempts, by judging the initial point from the previous failures, an orbit is at length obtained which is exactly periodic. There is practically one unknown constant and this has to be determined in each case by the trial and error method just explained. The devices for performing this work accurately, so that errors shall not accumulate, are ingenious.

The determination of the stability is a more difficult question, since it involves the making of an harmonic analysis of the orbit, and the computation of a determinant of many rows and columns, but it is essential and is carried out in each case with complete success. Many orbits belonging to several different families are found. This is pioneer work into a region practically untouched previously and has opened the way to further investigation. In a later paper improved methods and some additional orbits are given.

The remaining papers in the two volumes cover a variety of subjects and several of them are highly suggestive. They include astronomical and cosmogonic investigations, mechanical devices, thrust of a mass of sand, the formation of ripple marks in sand, the treatment of observations, and two statistical papers (1875) on the effects of marriage between first cousins. A detailed mention of them would take us too far. There are also three addresses which one is glad to have in permanent and convenient form, as well as the English version of "The Tides" written for the Encyclopädie der mathematischen Wissenschaften.

The care which the author has bestowed on the editing of his papers has added greatly to the value of the collection. Besides the correction of small errors, calculations have occasionally been repeated and in the cases where he considers that his work has been in error, such portions are omitted and an extract or even a whole paper by another writer who may have furnished the correction is inserted. One is struck too by the generous and full acknowledgment which is given to those from whose papers, letters, or remarks the

author has derived any ideas or assistance. The fourth volume completes the list of papers published up to the end of 1910: we hope that as further material accumulates it may be cast into the same form.

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MATHEMATICAL ECONOMICS.

Manuel d'Economie politique. Par VILFREDO PARETO. Paris, Giard et Brière, 1909. 695 pp.

IN the year 1906 Pareto's Manuale di Economia politica con una Introduzione alla Scienza sociale was published at Milan. Three years later a French translation by Alfred Bonnet, with revisions by the author, appeared as volume 38 in the Bibliothèque internationale d'Economie politique. This work, although written by the economist who has most insistently stood for the scientific mathematical method in economics, is not itself primarily mathematical except in spirit. The text to be sure, uses curves freely, is replete with logical keenness, and draws considerably upon mathematical language and upon mechanical, or even thermodynamical, analogy in discussing and illustrating economic equilibrium; but the strictly mathematical treatment, which might be technical to the point of causing some of the world's best literary economists insuperable difficulties, has in every instance been relegated to the long appendix of 133 pages. This arrangement has also been adopted by Irving Fisher in his recent work on The Purchasing Power of Money. So long as only a few students of political economics acquire the necessary knowledge of calculus such a segregation must remain inevitable.

In the first place, as there seems to be no very widespread notion among mathematicians, perhaps even among economists, as to what mathematical economics is and does, it may be well to define a little.

It is clear that the individual as a social unit and the state as a social aggregate require a certain modicum of mathematics, some arithmetic and algebra, to conduct their affairs. Under this head would fall the theory of interest, simple and compound, matters of discount and amortization, and, if

lotteries hold a prominent place in raising moneys, as in some states, questions of probability must be added. As the state becomes more highly organized and more interested in the scientific analysis of its life, there appears an urgent necessity for various statistical information, and this can be properly obtained, reduced, correlated, and interpreted only when the guiding spirits in the work have the necessary mathematical training in the theory of statistics. (Figures may not lie, but statistics compiled unscientifically and analyzed incompetently are almost sure to be misleading,* and when this condition is unnecessarily chronic the so-called statisticians may well be called liars.) The dependence of insurance of various kinds on statistical information and the very great place which insurance occupies in the modern state, albeit often controlled by private corporations instead of by the government, makes these theories of paramount importance to our social life.

As all these subjects of study are generally treated at our universities by the departments of political economy, if indeed they are treated at all, it might be natural to classify them as mathematical economics. And one doubtless recalls that M. Cantor adopts (from Petty?) the title Politische Arithmetik for his popular little text on such matters. But the term mathematical economics as Pareto† uses it, and as it is generally used, implies something very different, something much more akin to theoretical economics, just as the term mathematical physics does not denote the use of mathematics to reduce physical observations or even to give the theory of special physical apparatus so much as it connotes general theoretical physics. It is evident to every physicist who lifts his nose from the grindstone of his lengthy experiments, who frees and rests his mind from the ofttimes exasperatingly human perversities of his instruments, that it is only with the aid of mathematics that the salient facts of physics can be adequately set forth and correctly correlated, the underlying concepts and suppositions critically analyzed and properly refined. It is evident that notwithstanding the magnificent genius of Faraday, electromagnetism needed

* Those who have followed critically the government crops reports, emanating from the Department of Agriculture, have not failed at times to complain that better and plainer results might somehow be obtained.

† We would not imply that Pareto has not been interested in statistical questions; his researches in these lines are well known.

its Maxwell and its Lorentz, that howsoever wondrously Regnault might measure, thermodynamics needed its Gibbs and its Planck.

All this is true of economics, but here it is not true that every economist sees it. So intricate are the interrelations of economic phenomena, so hopeless are any extensive experimental methods, so fragmentary are as yet the desirable statistics, and so far removed from the possibility of accurate quantitative formulation does the subject seem to be, that perhaps the majority of economists, doing their best to grope their way through a fog of details and a mist of prejudices, feel it to be impossible, or at best a useless work of supererogation, to introduce mathematical methods into economics. Now, to be sure, if economists expect of the application of the mathematical method any extensive concrete numerical results, and it is to be feared that like other non-mathematicians all too many of them think of mathematics as merely an arithmetical science, they are bound to be disappointed and to find a paucity of results in the works of the few of their colleagues who use that method. But they should rather learn, as the mathematicians among them know full well, that mathematics is much broader, that it has an abstract quantitative (or even qualitative) side, that it deals with relations as well as numbers, and that it has been defined as the science which draws necessary conclusions.*

There is no doubt that economists draw conclusions. Of these some are tenable, some are not. A few of those tenable are necessary, many are not. Indeed it is inevitable, and for a long time will remain so, that economics, especially in its applications, should be an art, and that its conclusions, far from being necessary, should largely be a matter of the opinion of the individual economist. But just because the way is dark with detail and shrouded in prejudice and impossible of complete clarification, is that any reason why any one should scorn whatever illumination mathematics can bring, whatever precision of statement it may suggest, whatever necessary conclusions it offers? If one knows precisely what his premises are and just how much he can draw from them, is he not the better off for exercising his judgment when strict deduction is

* For a discussion of what mathematics is, see M. Bôcher, "The fundamental conceptions and methods of mathematics," in this BULLETIN, vol. 11, pp. 115–135.

no longer possible? So it seems to us, and we must admit that when we read anything that Pareto writes, whether a short article in a popular review or an essay* in the mathematical encyclopedia or a vast manual of political economy, we are agreeably impressed not alone with the cogency of the proofs, but even more with the soundness of the judgments.

Two quotations may be of interest as showing how vitally economists differ as to the use of mathematics in their field.

"Mathematics has influenced the form and the terminology of the science, and has sometimes been useful in analysis; but mathematical methods of reasoning, in their application to economics, while possessing a certain fascination, are of very doubtful utility."†

"The effort of the economist is to *see*, to picture the interplay of economic elements. The more clearly cut these elements appear in his vision, the better; the more elements he can grasp and hold in his mind at once, the better. The economic world is a misty region. The first explorers used unaided vision. Mathematics is the lantern by which what before was dimly visible now looms up in firm, bold outlines. The old phantasmagoria disappear. We see better. We also see further."‡

* We use this word advisedly; for it does not seem that Pareto's 50-page article (with apparently more to follow) in the new, French, edition of the mathematical encyclopedia is other than an essay, excellent in itself, but as an encyclopedia article on mathematical economics decidedly inferior to his earlier 25-page presentation of the same subject in the German edition. However, the whole encyclopedia, whether German or French edition, seems of late to have run riotously and fruitlessly to leaves.

† From the Encyclopedia Britannica, 11th edition, vol. 8, p. 902, in the long article on Economics which is signed W. A. S. H. As this signature is not reproduced in the key at the beginning of the volume, it is difficult to identify the author, but as this one sentence is his only reference to the vast literature of mathematical economics we are forced to believe that the editors of the new edition of the Britannica have in this instance, as in some others we have observed, let out an important subject to a man who, however eminent he may be, has not that complete command of his subject and that full sympathy with all its phases which are necessary and which alone bring competence for writing with the poise, dignity, and authority appropriate to a great encyclopedia. We should have had much more confidence in an article signed F. Y. E. On the other hand we must admit that W. A. S. H. is, in comparison with some others, extremely inoffensive and even gracious toward his mathematical confrères.

‡ From p. 119 of Irving Fisher's "Mathematical investigations in the theory of values and prices," *Transactions of the Connecticut Academy*, vol. 9 (1892), pp. 1–124. We cannot cite this work without commending it to all for the clearness and precision with which the economic problem is stated, for the keen appreciation of the limitations, as well as the advantages, of the use of mathematics in economics or other sciences, and

The foregoing paragraphs may seem rather remote from a review of the book under discussion; but they are not. For one of the things which the author makes clearest, by constant reference to it at all stages of his work, is the scientific attitude which he is taking, the attitude which he believes should be taken toward theoretical economics. He nowhere claims to do the impossible, he does not desire to overrate the usability of mathematics in applied economics, but he does insist upon an analysis, as thorough as possible, of the underlying foundations of economic theory. He is even willing to minimize the value of mathematics in the discussion of the simpler cases, where the reasoning may readily be carried out in words,* for the sake of emphasizing the fact that it is the interdependence of the totality of economic phenomena which brings in the complications necessitating the introduction of mathematics.

Let us begin now with the appendix. We consider an idealized individual, the homo œconomicus, who lives in the presence of and is possessed of certain goods and whose actions will result from his desire to obtain the greatest pleasure from the appropriation or "consumption" of the goods. If he is in possession of given amounts $x, y, \cdots$ of different goods, he will generally be equally satisfied, neither more nor less, by certain other combinations of amounts. If his possessions be represented by a point in a space of as many dimensions as the number of goods considered, the totality of the points which represent equal satisfaction will constitute a surface of one dimension less in that space. This surface is a surface of indifference, that is, of equal desire or pleasure. By considering that he could have a choice between some different combination and its totality of equivalents, another surface could be drawn. Thus the space is ruled off with a family of

for the critical discussion of the case against mathematics as presented by its opponents. The readers of Pareto's manual and of Fisher's monograph cannot fail to remark the similarity between the scientific points of view of the two authors.

* For instance on p. 557 he says: Nous estimons que l'usage des mathématiques, pour des problèmes du genre de celui d'un individu et de deux, ou même de plusieurs biens, ne donne pas des résultats dont l'importance puisse se comparer à ceux que l'on obtient dans les cas de l'équilibre économique général. A notre avis c'est l'interdépendance des phénomènes économiques qui nous oblige à faire usage de la logique mathématique. Cette manière de voir peut être bonne ou mauvaise; mais en tout cas elle ne doit pas être confondue avec celle des économistes qui établissent des théories en négligeant précisément cette interdépendance.

surfaces of indifference

$$f(x, y, \cdots; I) = 0 \quad \text{or} \quad I = \psi(x, y, \cdots) \quad \text{or} \quad I = F[\psi(x, y, \cdots)],$$

where the parameter I may be taken as the index of pleasure or ophelimity.*

We may now pass by differentiation to the equation

$$dI = F'\psi_x dx + F'\psi_y dy + \cdots = 0,$$

which is the differential equation of the surfaces of indifference. This equation merits a comment. The negative partial derivatives such as

$$-\left(\frac{dy}{dx}\right)_{y,\ldots} = +\frac{\psi_x}{\psi_y},$$

which it defines, express the rate at which one "good" will be indifferently exchanged for another. This, however, is nothing but the price of one good relatively to another. Hence the ratios

$$\psi_x : \psi_y : \cdots = p_x : p_y : \cdots$$

are the price ratios p relative to a fixed good as base. Be it observed, however, that these are not in general the actual market prices, but the prices which the given individual would himself establish. Moreover, the actual terms like $F'\psi_x = I_x$ are the rates of increase of I on the marginal amounts dx, and so forth; that is, they are the marginal ophelimities or utilities, which are so fundamental in economics.

Now conversely, one may estimate the price ratios of indifferent exchanges and may thus set up a differential equation

$$\varphi_x dx + \varphi_y dy + \cdots = 0$$

of which the integral would be I or a function of I. This, however, implies that such an integral exists. The discussion of the existence of this integral was suggested as a necessity by Volterra in commenting on the Italian edition of the manual.†

* The functional symbol F may be taken more or less arbitrarily according to the scale on which I is estimated; for although the value of I may be distinguished from a greater or a smaller value, the estimation of I is hardly quantitative.

† The same suggestion was made already in 1892 by Fisher, loc. cit., p. 88, on mathematical grounds; but I believe that considerations indicated by Gibbs have since led him to abandon the suggestion.

The author now gives a careful treatment of the question, which is necessarily somewhat complicated by the every-day fact that the consumption of certain amounts, dx, dy, $\cdots$ of the various goods may yield very different increments of pleasure according to the order in which they are consumed—not many would care to reverse the order of their dinner menu. (Pareto, page 251).

Now this reference to a menu is exceedingly liable to cause confusion; for the order of consumption which is in question as regards the integrability of the above differential is merely a conceptual order or curve in the space of the goods x, y, $\cdots$, whereas the order of consumption which is involved in the menu is a time order which either has nothing to do with the other order or has some very fundamental and subtle and neglected relation to the whole question of utility. Suppose that we could integrate the equation and thus obtain an index of pleasure; this index ought, it would seem, to apply either at a given instant (and to contain the time as a parameter, owing to the fact that at different times the price ratios and integrating factor are different), or it ought to apply to an interval of time such as a day or a year during which conditions were supposed to remain the same (and the time would then not enter at all). Fisher in his memoir distinctly takes the latter point of view, and indicates its advantages in smoothing out capricious elements, whims instead of well balanced desires. This is the time-average way of obtaining definite results. But the homo œconomicus may well be considered an idealized averaged individual free from caprice. This is the ensemble-average way of getting definiteness. Pareto seems to us to take this point of view. (Students of statistical mechanics are acquainted with the two types of average and with their essential identity for many problems.) But in neither case does the time enter into the determination of the equilibrium or of the conceptual path followed to attain the point of maximum pleasure—unless, indeed, the whole analogy with mechanics and thermodynamics is to break down in a vital point. We must remember that in statics the time element is ignored and that in thermodynamics changes are supposed to take place infinitely slowly. The difference in the increase of pleasure which a person would experience in eating his dinner in inverse as opposed to direct order cannot, we believe, be logically taken into account, and we are inclined to think

that the whole discussion of the order of integration, a point that naturally occurs to the mathematician, can be thrown out of court by the economist. For on the ensemble-average basis with an instantaneous comparison of the desirability of increments of different goods, there is no period of time during which they can be consumed in temporal order, the "consumption" is mere conceptual appropriation; whereas on the time-average basis the consumer is free to consume in the temporal order he chooses. The author in places comes pretty near saying this himself; he might have decided to say it if he had observed that to eat one's dinner in the proper order with too little leisure or with too lengthy waits between the courses would be quite as bad as to have to eat it in the wrong order, for we suppose he would hardly desire to include the intervals of time themselves as consumed goods so as still to have different orders.

We have spoken of the exchange ratios or price ratios which the given individual would regard as fitting his desires. These are not in general the market prices, or proportional to them. Hence the homo œconomicus will avail himself of this divergence to make exchanges which will increase his value of I, and he will pursue these exchanges until his private exchange ratios become identical with those of the market. He will thus come into equilibrium with the market with those particular amounts $x, y, \cdots$ of goods which will maximize I subject to the imposed constraints. For example, if he starts with the amounts $x_0, y_0, \cdots$ and if the market prices are constant,* his exchanges must satisfy the equation

$$(x - x_0)p_x + (y - y_0)p_y + \cdots = 0,$$

and his price ratios must satisfy the $n - 1$ equations

$$\varphi_x : \varphi_y : \cdots = p_x : p_y : \cdots.$$

Thus there are just n equations wherefrom to determine the final or equilibrium values of the n quantities $x, y, \cdots$. The problem is determinate.

* Although in the last analysis it is the desires of all which regulate the prices in the market, it is permissible as a first approximation to many phenomena to regard these prices as constant for any one individual; his purchases and sales are insufficient materially to change the market quotations.

This simple problem in the determination of a point of equilibrium is typical of the various more complicated problems of the same sort. The analysis really amounts to formulating the equations, counting the number of unknowns, and checking it with the number of equations. To many modern mathematicians the fear that the equations might be either redundant or incompatible would probably be so strong as to deter them from seeing much of value in the analysis. But it should be remembered that not so very long ago the method of counting constants was widely used in pure mathematics even though the science was then much more highly developed toward arithmetic equations than is now the case with economics. Moreover, in a physical science the question of rigor is very different from that in mathematics; to be ultra-rigorous mathematically may be to be infra-rigorous physically. To throw out Gibbs's phase rule because its proof, being essentially a count of constants, is no proof at all would be equally good mathematics and equally bad physics. On the other hand, setting up exact mathematical relations, compatible and uniquely soluble, over the whole range of variation of the variables might be a wonderful mathematical tour de force while being viciously misleading physics in the neighborhood of certain critical points where a slight change of the variables introduces such wide variations in the functions as to make the problem just as indeterminate physically as it has become determinate mathematically.

It would not do, however, to give the impression that the use of analysis in economics is restricted to counting constants. It is necessary to study the general properties of the surfaces of indifference, and this leads to the quadratic differential form for the second differential d^2I. The law of demand and supply, in the simple case of a single individual in relation to the whole market, requires partial derivatives and transformations among them which are well comparable to those required in thermodynamic theory. If space permitted, we should be glad to stop at this point and sketch the proof of the law that the demand decreases as the price rises. It would perhaps be still more indicative of the usefulness of mathematics if we could run through the author's analysis of Marshall's hypothesis that for small variations of price and of quantities the marginal utility of money could be treated as constant; for Pareto shows by an extended series of reductiones ad absurda,

which apparently would be difficult to establish without some use of mathematics, that the hypothesis is not tenable. Further on, in the discussion of production, which is probably the most complicated in the whole appendix, it becomes necessary to maximize integrals, and the author is therefore led to introduce into his analysis the ordinary algorisms of the calculus of variations.

It is thus seen that it is not merely the mathematics of a first course in calculus which may be used to advantage in economic theory. Interesting as it would be to pursue these matters in detail, we feel that we must come back to the main text and offer some short account of it. In the first chapter on general principles the author first analyzes the different aims which may lie behind the study of political economy and carefully delimits himself to the single one of developing theoretical economics. He then turns his attention to discussing the relation of a theory to the actual phenomena of which it is the theory. Although he is writing with his own subject in view, this introductory essay is conceived and carried out in such a broad spirit that it may serve almost equally well as a sort of prolegomenon to the study of any scientific theory, and as such it can be recommended for general reading.

Chapter II, an introduction to social science, is delightful reading and forms a sort of amplification of the preceding chapter upon the social and moral side. Honest and manly in its views, pithy and pungent in its epigram, it calls to mind the direct diction and unrelenting exposition of W. G. Sumner's Folkways. Pareto, however, did not make use of, and perhaps did not have at hand, the mass of detailed references to folkways which Sumner adduced to drive in and rivet in his opinions. For this reason those who are especially interested in this chapter should supplement their study of it by reading the Folkways. Indeed if economics is a misty region under a pall of prejudices, where fierce war is often waged around the banners of certain words—trusts, protection, supply and demand, single tax—of the meaning of which the combatants have not always a very clear perception, even more is this true of social science, and one who would really attain a free and independent judgment in this field has to exercise the greatest care. One of the most interesting observations of the author is (page 122) that there is an antagonism between

the conditions of action and of knowledge, that faith alone drives men to action, and that it is not for the good of society that very many persons should be occupied with the scientific study of social questions.*

It is the next four chapters, Notion générale de l'équilibre économique, Les goûts, Les obstacles, L'équilibre économique, which contain for the most part that portion of the text which is supplemented by the mathematical appendix, and which is most scientific in its development. It is this part which seems so different from the older treatises on economics, not only on account of the generous use of curves as in a number of modern texts, but particularly because of the great care with which the problems arising are divided into separate sub-problems and the exhaustive manner in which the various sub-cases are treated. We have tried to give an idea of economic equilibrium in a simple case; for any idea of the multifarious distinctions which arise in the presence of various degrees of free competition and monopoly we must refer to the text. But to show how conscientious the author is in exhibiting real as well as theoretical phenomena, we will cite an instance or two. It is a statement often found in texts on economics that the more we have of a thing, the less valuable to us are equicrescent increments of it; that is, the marginal utility of anything decreases with an increasing amount of it. And many an author states this as a general law. Pareto, however, makes clear the fact that there are exceptions. It does no harm to amplify the matter even further, to point out that in this life our desires consist of a lot of little ones and a few big ones, that to satisfy the latter large increments of certain goods (say, of money) are necessary, and that therefore the curve of marginal utility of money consists in reality of long slow descents (in the intervals where the little desires are better and better satisfied, but the satisfaction of one of the large ones seems hopelessly remote) interrupted by sharp ascents (in the intervals where the fulfilment of one of the greater desires seems imminent). In another place the author, when discussing the proposition that the returns on

* A footnote appended to these remarks may be of interest. Par example le livre de M. Roosevelt, American Ideals, pourra peut-être servir à pousser à l'action les citoyens des Etats-Unis, mais assurément il n'ajoute rien à nos connaissances, et sa valeur scientifique est très voisine de zéro. Pareto never minces matters in the expression of his opinion, and to a careless or prejudiced reader might therefore appear unjust or bitter when he meant to be only frank.

capital differently invested ought in a state of equilibrium to be the same, cites some very interesting statistics concerning Belgian corporations. It appears that from 1873 to 1887 there were founded 1,088 companies with a paid-in capital of 1,493 millions of francs. Up to 1901 the sum of 515 millions had been paid back under liquidation, and the remaining companies were earning 56 millions a year. This is only 5.7 per cent. on the 978 millions remaining in the original investment. Although the figures are only rough, it is certain that the return on the 1,088 enterprises could not have much exceeded the current rate of interest. In this connection let us remark that some student statistician looking for a thesis might consider the problem of calculating as accurately as possible the rate of return on capital variously invested in this country.

Chapter VII is on various topics connected with the population—the distribution of wealth, relations between economic conditions and the population, malthusianism, stability and selection, and so on. In the next chapter the author takes up capital, and treats savings, interest, exchanges, international commerce, bimetalism, and the like. His definition of capital appears to us considerably less definite than that of Irving Fisher in his Capital and Income and Rate of Interest.* Indeed it may be safe to hazard the guess that Pareto, who seems to esteem Fisher's work very highly, might have made considerable use of it if only it had appeared early enough to have been easily available for the Manuel. It goes without saying that when treating such subjects as population and capital, the author is merging into the realms of applied economics where the argument has to run somewhat differently from its previous course.

In the final chapter, Le phénomène économique concret, we have, among many other things, a delightful essay on present conditions and tendencies in our economic or social life. To any one who is perplexed at our rapid social evolution, who is confused in his ideas of trusts or of labor unions, who is bewildered at the way in which the rising costs of government are bleeding the bourgeoisie for the support of the lower classes, to any one who wonders where we are and whither we are going, no better reading can be suggested than this chapter of Pareto's; it is short enough to keep the attention, virile

* Reviewed in this BULLETIN, vol. 15, p. 169.

enough to claim the admiration, of the most nonchalant reader. Much as the author's scientific economics appeals to us, we must say that his sturdy vision into society and social evolution are even more to our liking; and there can be little doubt that for many of us, mathematicians or otherwise, the most inspired portions of Pareto's Manuel will be the societology of Chapters II and IX.

In closing this review it may be only proper to apologize a bit for its tardiness. At the same time we may perhaps point out that it is now while we are in the midst of modifying constitutions, whether actually or under various disguises such as preferential primaries or judicial rule of reason, while we are attacking or about to attack the formerly sacred institution of private property, under many a pretext other than the real one that might makes right, now while we in America are perhaps more than ever conscious of our present or impending social evolution is the time when we may derive the most benefit from having our attention called to Pareto's observations on the ills that beset us. As for the mathematical economics, that is still good, and still will be good at any time when we may settle down to study it.

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NOTES.

THE programme of the Fifth international congress of mathematicians at Cambridge, England, begins on Wednesday evening, August 21, with a reception of the members in the Hall of St. John's College. The opening meeting will be held on Thursday morning and the afternoon will be devoted to organisation, election of officers, and two of the special lectures. On the following days sectional meetings will be held in the morning and lectures will be delivered in the afternoon. The congress will divide into four sections: I. Arithmetic, algebra, analysis; II. Geometry; III. Mechanics, physical mathematics, applications; IV. Philosophical, historical, and didactic questions. In connection with Section IV three discussions inaugurated by the International commission on the teaching of mathematics will be held. The sections will be further subdivided as the number of papers may require.

The following special lectures are announced: M. BÔCHER: "Boundary problems in one dimension." E. BOREL: "Définition et domaine d'existence des fonctions monogènes uniformes." E. W. BROWN: "Periodicity in the solar system." F. ENRIQUES: "I problemi relativi ai principii della geometria." B. GALITZIN: "The principles of instrumental seismology." E. LANDAU: "Gelöste und ungelöste Probleme aus der Theorie der Primzahlverteilung und der Riemannschen Zeta-funktion." J. LARMOR: "The dynamics of radiation." W. H. WHITE: "The place of mathematics in engineering practice."

Members of the congress unaccompanied by ladies can be accommodated in the colleges at a fixed payment of 6*s*. per day for rooms, attendance, breakfast, and dinner. Luncheon will be provided at 1*s*. 6*d*. Ladies will be received at Newnham College at 6*s*. per day for rooms, attendance, and meals. Members desiring accommodation in the Colleges are requested to notify the general secretary, Professor E. W. Hobson, Christ's College, before August 1. The fee for membership in the congress is £1. The family of a member are entitled to the privileges of membership, except the receipt of a copy of the Proceedings, on payment of 12*s*. each.

AT the meeting of the London mathematical society held on April 17 the following papers were read: By H. S. Carslaw "An application of the theory of integral equations to the equation $\nabla^2 u + k^2 u = 0$"; by A. Cunningham, "On Mersenne's numbers."

THE following university courses in mathematics are announced for the year 1912–1913:

UNIVERSITY OF CHICAGO. All courses are four hours a week.—By Professor E. H. MOORE: Integral equations in general analysis; General seminar on mechanical quadrature, continued fractions, and boundary problems (throughout the year).—By Professor G. W. MYERS: History of mathematics (winter).—By Professor L. E. DICKSON: Theory of invariants (autumn and winter); Theory of numbers (winter); Theory of equations and linear algebras (spring).—By Professor J. W. A. YOUNG: Limits and series (winter).—By Professor H. E. SLAUGHT: Differential equations (autumn).—By Professor G. A. BLISS: Theory of functions (autumn); Definite integrals and

abelian integrals (winter); Hyperelliptic functions (spring).—By Professor E. J. WILCZYNSKI: Selected topics in geometry (autumn and winter); Projective differential geometry (spring).—By Professor A. C. LUNN: Graphical analysis and theory of attraction and potential (autumn); Fourier series and Bessel's functions and vector analysis (spring).

CLARK UNIVERSITY.—By Professor W. E. STORY: Analytic geometry of higher plane curves, higher surfaces, and twisted curves, three hours; Calculus of operations and finite differences, three hours; Theory of errors, three hours first half-year; Infinitesimal geometry, three hours second half-year; Seminar.—By Professor H. TABER: Theory of functions, five hours; Integral equations, two hours first half-year; Hypercomplex number systems, two hours second half-year; Seminar.—By M. DE PEROTT: Theory of numbers, two hours first half-year; Abelian integrals, two hours second half-year.

HARVARD UNIVERSITY.—By Professor W. E. BYERLY: Advanced calculus, three hours; Dynamics of a rigid body, three hours; Trigonometric series, introduction to spherical harmonics and the potential function, three hours, with Professor B. O. PEIRCE.—By Professor W. F. OSGOOD: Advanced algebra, three hours (second half year); Theory of functions, second course, three hours.—By Professor M. BÔCHER: Ordinary linear differential equations, three hours.—By Professor C. L. BOUTON: Elementary theory of differential equations, three hours (first half year); Geometric transformations, three hours.—By Professor E. V. HUNTINGTON: Fundamental concepts of mathematics, three hours (second half year).—By Professor J. L. COOLIDGE: Introduction to modern geometry and modern algebra, three hours; Geometry of the circle, three hours.—By Professor G. D. BIRKHOFF: Theory of functions, first course, three hours; Calculus of variations, three hours (first half year).—By Dr. D. JACKSON: Infinite series and products, three hours (first half year); Definite integrals, three hours (second half year).

Various courses in reading and research are also offered on special topics, and Professors OSGOOD and BIRKHOFF will conduct a fortnightly seminar in the theory of functions.

UNIVERSITY OF ILLINOIS.—By Professor E. J. TOWNSEND: Complex variables, three hours.—By Professor G. A. MILLER:

Elementary group theory, three hours.—By Professor H. L. RIETZ: Actuarial theory, three hours (first term).—By Professor C. H. SISAM: Differential geometry, three hours.—By Professor J. B. SHAW: Fourier series, three hours.—By Professor A. EMCH: Elliptic functions, three hours.—By Dr. A. R. CRATHORNE: Linear differential equations, three hours.—By Dr. R. L. BÖRGER: Modern algebra, three hours.—By Dr. E. B. LYTLE: History of mathematics, three hours.

INDIANA UNIVERSITY.—By Professor S. C. DAVISSON: Ordinary differential equations (a, w), three hours; Fourier's series (s), three hours; Theory of functions (a, w, s), two hours.—By Professor D. A. ROTHROCK: Advanced calculus (a, w, s), three hours; Higher geometry (a, w), three hours.—By Professor U. S. HANNA: Theory of errors (a), three hours; Substitution groups and Galois theory (w, s), three hours.—By Professor R. D. CARMICHAEL: Functions of an infinite number of variables (a, w), three hours; Partial differential equations (a, w), three hours; Theory of numbers (s), five hours; Seminar in difference equations (a, w, s), two hours.
(a, w, s = autumn, winter, spring quarters.)

JOHNS HOPKINS UNIVERSITY.—By Professor F. MORLEY: Higher geometry, two hours; Dynamics, two hours (second term); Seminar, two hours.—By Professor A. B. COBLE: Theory of correspondences, two hours; Theory of probabilities, two hours (second term).—By Professor A. COHEN: Theory of functions, two hours; Differential equations, two hours (first term); Theory of numbers, two hours (second term).—By Mr. H. BATEMAN: Integral equations, two hours.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.—By Professor H. W. TYLER: Elements of the theory of functions, two hours.—By Professor F. S. WOODS: Partial differential equations, two hours.—By Professor F. H. BAILEY: Fourier series, two hours.—By Professor E. B. WILSON: Advanced calculus and differential equations, three hours; Hydrodynamics, one hour; Theoretical mechanics, two hours; Mathematics of wireless telegraphy, two hours.—By DR. H. B. PHILLIPS: Thermodynamics, two hours.—By DR. F. L. HITCHCOCK: Curve tracing, two hours; Elementary theory of the potential function, two hours.

PRINCETON UNIVERSITY.—By Professor H. D. THOMPSON: Analytic geometry, three hours; Infinitesimal geometry, three

hours.—By Professor L. P. EISENHART: Differential geometry, three hours; Mechanics, three hours.—By Professor O. VEBLEN: Algebra, three hours; Seminar, three hours.—By Professor J. G. HUN: Analytic projective geometry (second term), three hours.—By Professor E. SWIFT: Calculus of variations (second term), three hours.—By Professor J. H. McL. WEDDERBURN: Theory of functions of a complex variable, I (first term), three hours.

THE following courses are announced for the summer session of 1912:

UNIVERSITY OF CHICAGO.—All courses are four hours a week. —By Professor E. H. MOORE: Integral equations, first term; General analysis, first term.—By Professor E. J. WILCZYNSKI: Synoptic course in advanced mathematics; Higher geometry. —By Professor A. C. LUNN: Graphical analysis; Units and dimensions; Vector analysis.—By Professor E. R. HEDRICK: Definite integrals; Theory of functions of a complex variable. —By Professor G. A. MILLER: Finite groups.

COLUMBIA UNIVERSITY.—By Professor JAMES MACLAY: Theory of functions of a complex variable.—By Professor M. W. HASKELL: Modern analytic geometry.—By Professor W B. FITE: Theory of groups of finite order.

CORNELL UNIVERSITY.—All courses are five times a week. —By Professor V. SNYDER: Projective geometry.—By Professor W. B. CARVER: Teachers' course in geometry.—By Dr. C. F. CRAIG: Differential equations.—By Dr. F. W OWENS: Teachers' course in algebra.

INDIANA UNIVERSITY.—By Professor S. C. DAVISSON: Ordinary differential equations, five hours, first half-quarter.— By Professor D. A. ROTHROCK: Advanced calculus, five hours; Higher plane curves, three hours.—By Professor R. D. CARMICHAEL: Linear differential equations, five hours; Infinitesimal analysis, three hours; Difference equations, two hours.

PROFESSOR G. KOWALEWSKI, of the technical school at Prague, has accepted a professorship of mathematics at the German University of Prague.

DR. R. COURANT has been appointed docent in mathematics at the University of Göttingen.

DR. HOSTINSKY has been appointed docent in mathematics at the Bohemian University of Prague.

DR. F. PFEIFFER has been appointed docent in mathematics at the technical school at Danzig.

PROFESSOR P. APPELL, of the University of Paris, has been elected a corresponding member of the academy of sciences of St. Petersburg.

PROFESSOR E. CZUBER, of the University of Vienna, has been elected to membership in the academy of sciences of Halle.

PROFESSOR A. DEMOULIN, of the University of Ghent, has been elected a corresponding member of the royal society of Liège.

PROFESSOR G. A. MILLER, of the University of Illinois, has recently been elected corresponding member of the Sociedad matemática Española of Madrid.

PROFESSOR P. P. BOYD, of Hanover College, Indiana, has accepted a professorship of mathematics at the Kentucky State University.

PROFESSOR H. B. FINE, of Princeton University, has been granted leave of absence during the academic year 1912–1913, to study and travel in Europe.

MR. W. W. DENTON and DR. M. W. CHITTENDEN have been appointed instructors in mathematics at the University of Illinois.

MR. C. A. BARNHART, of the University of Illinois, has been appointed professor of mathematics at Carthage College, Carthage, Illinois.

DR. R. K. MORLEY, of the University of Illinois, has been appointed assistant professor of mathematics at the Worcester Polytechnic Institute.

DR. G. F. MCEWEN, of the University of Illinois, has accepted a position as hydrographer for the biological station of the University of California.

MR. A. S. GALAJIKIAN, of Cornell University, has been appointed assistant professor of mathematics at the University of the Philippines.

MR. L. C. COX, of State College, Pennsylvania, has been appointed instructor in mathematics at Purdue University.

DR. J. R. CONNER, of the Johns Hopkins University, has been appointed associate in mathematics at Bryn Mawr College.

DR. J. I. TRACEY has been appointed instructor in mathematics in the academic department of Yale University.

DR. R. M. WINGER has been appointed instructor in mathematics at the University of Illinois.

MR. H. BATEMAN, of Bryn Mawr College, has accepted a teaching scholarship at the Johns Hopkins University.

MR. W. V. LOVITT, of the University of Washington, has been appointed instructor in mathematics at Harvard University; Mr. A. L. MILLER has been advanced to an instructorship in mathematics.

PROFESSOR A. E. HAYNES, of the University of Minnesota, has retired after thirty-eight years continuous service as a teacher.

PROFESSOR S. W. SHATTUCK, of the University of Illinois, will retire at the end of the present academic year. He has been connected with the department of mathematics since the founding of the University in 1868.

PROFESSOR M. C. ARZELÀ, of the University of Bologna, died at St. Stephen di Magra, March 16, 1912, at the age of 65 years. He was a member of the Academy dei Lincei and of the national Italian society of sciences.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ADHÉMAR (R. d'). Leçons sur les principes de l'analyse. Tome 1: Séries. Déterminants. Intégrales. Potentiels. Equations intégrales. Equations différentielles et fonctionelles. Paris, Gauthier-Villars, 1912. 8vo. 6+324 pp. Fr. 10.00

BACHELIER (L.). Calculs des probabilités. Tome 1. Paris, Gauthier-Villars, 1912. 4to. 7+518 pp. Fr. 25.00

CALVET-AZAL. Essai sur la notion de quantité imaginaire. Paris, Gauthier-Villars, 1912. 8vo. 48 pp.

CERVALLO (E.). Le calcul des probabilités et ses applications. Paris, Gauthier-Villars, 1912. 8vo. 9+171 pp. Fr. 6.50

LIEBMANN (H.). See MARKOFF (A. A.).

MANGOLDT (H. v.). Einführung in die höhere Mathematik. 2ter Band: Differentialrechnung. Leipzig, Hirzel, 1912. 8vo. M. 15.40

MANNING (H. P.). Irrational numbers and their representation by sequences and series. New York, Wiley. 12mo. 6+123 pp. Cloth. $1.25

MARKOFF (A. A.). Wahrscheinlichkeitsrechnung. Nach der 2ten Auflage des russischen Werkes übersetzt von H. Liebmann. Leipzig, Teubner, 1912. 8vo. 7+318 pp. Cloth. M. 13.00

MEISSNER (O.) Wahrscheinlichkeitsrechnung nebst Anwendungen. (Mathematische Bibliothek, herausgegeben von W. Lietzmann und A. Witting.) Leipzig, Teubner, 1912, 8vo. 4+64 pp. M. 0.80

MÜHE (A.). Beweis des Fermatschen Prinzips. 2te, ergänzte Auflage. Berlin, Reimer, 1912. 8vo. 4 pp. M. 0.75

II. ELEMENTARY MATHEMATICS.

BARLOW. Table of squares, cubes, square roots. New York, Spon, 1911. 12mo. 200 pp. $1.50

BERTRAND (G.). Trattato di algebra elementare. Prima traduzione italiana, con note ed aggiunte di E. Betti. Nuova edizione, per cura di A. Socci. 15a impressione. Firenze, Le Monnier, 1912. 16mo. 7+531 pp. L. 3.00

BETTI (E.). See BERTRAND (G.).

BORCHARDT (W. G.). Junior algebra examples. London, Rivingtons, 1912. 8vo. 198 pp. 2s.

HENSING (K.). Geometrie. 3tes Heft. Leipzig, Hoffmann, 1911. 8vo. pp. 115–254. M. 1.60

HUNTINGTON (E. V.). Four place table of logarithms and trigonometric functions. Unabridged edition. London, Spon, 1912. 3s.

LESSER (O.). See SCHWAB (K.).

LINNICH (M.). See SCHWAB (K.).

MILNE (W. J.). Key to first year algebra. New York, American Book Co., 1912. 12mo. 288 pp. $0.80

ROPP (C.). Calculator and short-cut arithmetic. 10th edition, revised and enlarged. Chicago, Ropp, 1912. 8vo. 192 pp. $1.25

SALOMON (A.). Leçons d'algèbre. 7e édition. Paris, Vuibert, 1912. 16mo. 267 pp. Fr. 2.00

SCHMEHL (C.). Lehrbuch der Stereometrie für höhere Lehranstalten. Giessen, Roth, 1912. 8vo. 6+160 pp. M. 2.50

SCHWAB (K.) und LESSER (O.). Mathematisches Unterrichtswerk. Für höhere Mädchenschulen bearbeitet von M. Linnich. 1ter Teil. Wien, Tempsky, 1911. 8vo. 143 pp. Cloth. M. 2.00

——. Geometrie. Bearbeitet von M. Linnich. 2ter Teil. Leipzig, Freytag, 1912. 8vo. 184 pp. Cloth. M. 2.50

SOCCI (A.). See BERTRAND (G.).

TABLES de logarithmes à cinq décimales et pour les fonctions trigonométriques. Par F. I. C. Edition double: Division sexagésimale et division centésimale. Paris, Gigord, 1912. 16mo. 8+255 pp.

TESTI (G. M.). Elementi di matematica. Fascicolo II. 5a edizione corretta. Livorno, Giusti, 1912. 16mo. 7+78 pp. L. 0.80

WALLENTIN (F.). Maturitätsfragen aus der Mathematik. Auflösungen. 6te Auflage. Wien, Gerold, 1912. 8vo. 6+236 pp. Cloth. M. 4.50

III. APPLIED MATHEMATICS.

ABRAHAM (M.). Theorie der Elektrizität. 1ter Band: Einführung in die Maxwellsche Theorie der Elektrizität. Mit einem einleitenden Abschnitte von A. Föppl. 4te, umgearbeitete Auflage, herausgegeben von M. Abraham. Leipzig, Teubner, 1912. 8vo. 18+410 pp. Cloth. M. 11.00

ANNUAIRE pour l'an 1912, publié par le Bureau des Longitudes. Avec des notices scientifiques. Paris, Gauthier-Villars. Fr. 1.50

BAKKER (G.). Théorie de la couche capillaire des corps purs. Tome II: La couche capillaire des corps purs et ses phases homogènes et adjacentes. (Collection Scientia.) Paris, Gauthier-Villars, 1912. 8vo. 80 pp. Boards. Fr. 2.00

BIGOURDAIN (G.). See DELAMBRE (J. B. J.).

BUCHHOLZ (H.). See KLINKERFUES (W.).

CHAPMAN (S.). The kinetic theory of a gas constituted of spherically symmetrical molecules. London, Dulau, 1912. 4to. Sewed. 2s. 6d.

CHINI (M.). Corso speciale di matematiche, ad uso dei chimici e dei naturalisti. 2a edizione, con aggiunte e modificazioni. Livorno, Giusti, 1912. 8vo. 12+297 pp. L. 4.00

CRANDALL (C. L.). Textbook on geodesy and least squares. New York, Wiley. 8vo. 10 + 329 pp. Cloth. $3.00

DALLWITZ (W. v.). Wärmelehre in Theorie und Anwendung. 1ter Band: Wärmetheorie und ihre Beziehungen zur Technik und Physik. Berlin, Volckmann, 1912. 8vo. 18+331 pp. Cloth. M. 11.25

DELAMBRE (J. B. J.). Grandeur et figure de la terre. Ouvrage augmenté de notes, de cartes, et publié par les soins de G. Bigourdain. Paris, Gauthier-Villars, 1912. 8vo. 8+402 pp. Fr. 15.00

DELSOL (M. E.). Note sur le vol des oiseaux. Paris, Gauthier-Villars, 1911. 8vo. 22 pp. Fr. 1.00

DITTMANN (E.). Berechnung elektrischer Leitungsnetze. Strelitz, Hittenkofer, 1911. M. 5.00

ESPITALLIER et CHASSERIAUD (R.). Cours d'aviation. Livre I: Appareils d'aviation et propulseurs. Paris, Gauthier-Villars, 1911. 4to. 295 pp. Fr. 13.00

FÖPPL (A.). See ABRAHAM (M.).

FRISCHAUF (J.). See SOLDNER (J.).

GRAMONT (A. de). Essais d'aérodynamique. 2e série. Paris, Hachette, 1912. 4to. 115 pp. Fr. 3.50

GROSSMANN (H.). Agenda 1912 pour l'horlogerie, la bijouterie et la petite mécanique. 9e année. Paris, Gauthier-Villars, 1911. 32mo. 372 pp. Fr. 2.00

GUILHAUMON (J. B.). Astronomie et navigation suivies de la compensation des compas. 5e édition. Paris, Berger-Levrault, 1912. 8vo. 615 pp. Fr. 10.00

HAUCK (G.). Vorlesungen über darstellende Geometrie. Herausgegeben von A. Hauck. 1ter Band. Leipzig, Teubner, 1912. 8vo. 12+339 pp. Cloth. M. 12.00

HOSMER (G. L.). Azimuth. New York, Wiley, 1909. 16mo. 5 + 73 pp. Morocco. $1.00

INGALLS (J. M.). Interior ballistics. 3d edition. New York, Wiley, 1912. 8vo. 221 pp. Cloth. $2.50

JEANS (J. H.). The mathematical theory of electricity and magnetism. 2d edition. London, Cambridge University Press, 1911. 15s

KING (W. I.). The elements of statistical method. New York, Macmillan, 1912. 12mo. 245 pp. Cloth. $1.50

KLINKERFUES (W.). Theoretische Astronomie. Neubearbeitung von H. Buchholz. 3te, verbesserte und vermehrte Auflage. Braunschweig, Vieweg, 1912. 8vo. 38+1070+11 pp. Cloth. M. 50.00

KOENEN (M.). Grundzüge für die statische Berechnung der Beton- und Eisenbetonbauten. 4te, neubearbeitete Auflage. Berlin, Ernst, 1912. M. 2.00

KRIEMLER (C.). Einführung in die energetische Baustatik. Berün, Springer, 1911. M. 2.40

KÜSTER (F. W.). Logarithmische Rechentafeln für Chemiker, Pharmazeuten, Mediziner und Physiker. 12te, neuberechnete Auflage. Leipzig, Veit, 1912. 8vo. 107 pp. Cloth. M. 2.40

LONEY (S. L.). An elementary treatise on statics. London, Cambridge University Press, 1912. 8vo. 402 pp. 12s.

LUCAS (A.). See MASSENET (G.).

MASSENET (G.) et LUCAS (A.). Eléments de la théorie du navire. Paris, Challamel, 1912. 8vo. 6+120 pp.

MAYER (R.). Die Mechanik der Wärme. 2te Abhandlung. Herausgegeben von A. von Oettingen. (Ostwalds Klassiker Nr. 180.) Leipzig, Engelmann, 1911. M. 1.60

MEIER (K.). Mechanics of heating and ventilation, with charts for calculation and examples. New York, McGraw-Hill, 1912. 8vo. 9+161 pp. $5.00

MERBACH (P. A.). Studien über die Mechanik der magnetischen Erscheinungen. Hamburg, Hephaestos-Verlag, 1912. 8vo. 18 pp. M. 0.60

MERRIMAN (M.). Elements of mechanics. Forty lessons for beginners in mechanics. New York, Wiley. 12mo. 172 pp. Cloth. $1.00

——. Treatise on hydraulics. 9th edition, revised and reset. New York, Wiley, 1912. 8vo. 10 + 565 pp. Cloth. $4.00

—— and JACOBY (H. S.). A textbook on roofs and bridges. Part III: Bridge design. 5th edition, partly rewritten. New York, Wiley, 1912. 8vo. 8 + 414 pp. Cloth. $2.50

MILLER (E. F.) and others. Problems in thermodynamics and heat engineering. New York, Wiley, 1911. 8vo. 3+67 pp. $0.75

MÜLLER (E.). Technische Uebungsaufgaben für darstellende Geometrie. 4tes Heft. Wien, Deuticke, 1911. M. 1.25

NERNST (W.). See POLLITZER (F.).

OCAGNE (M. d'). See SECO DE LA GARZA (R.).

OETTINGEN (A. von). See MAYER (R.).

PARKER (G. W.). The elements of hydrostatics. London, Longmans, 1912. 8vo. 3s.

PERKINS (H. A.). An introduction to general thermodynamics. New York, Wiley, 1912. 12mo. 18+247 pp. Cloth. $1.50

PFARR (A.). Die Turbinen für Wasserkraftbetrieb. Ihre Theorie und Konstruktion. 2te, teilweise umgearbeitete und vermehrte Auflage. 2 Bände. Berlin, Springer, 1912. Cloth. M. 40.00

PLEMELJ (J.). Potentialtheoretische Untersuchungen. (Gekrönte Preisschrift.) Leipzig, Teubner, 1911. M. 6.00

POLLITZER (F.). Die Berechnung chemischer Affinitäten nach dem Nernstschen Wärmetheorem. Mit einem Vorwort von W. Nernst. Stuttgart, Enke, 1912. 8vo. 171 pp. M. 3.60

RIECKE (E.). Lehrbuch der Physik. 1ter Band: Mechanik, Molekularerscheinungen, Akustik und Optik. 5te, vermehrte und verbesserte Auflage. Leipzig, Veit, 1912. 8vo. 16+600 pp. Cloth. M. 13.00

RYAN (W. T.). Design of electrical machinery. In three volumes. Volume 1. New York, Wiley, 1912. 8vo. 9+109 pp. $1.50

SATTERLY (J.). Junior heat. London, Clive, 1912. 8vo. 192 pp. 2s.

SECO DE LA GARZA (R.). Les nomogrammes de l'ingénieur. Avec une préface de M. d'Ocagne. Paris, Gauthier-Villars, 1912. 8vo. 12+196 pp. Fr. 12.00

SELLANDER (H. W.). Lightning calculator for marine engineers. 2d edition, revised and enlarged. San Francisco, King's Bookstore, 1912. 16mo. 141 pp. Boards. $2.50

SOLDNER (J.). Theorie der Landesvermessung. (1810.) Herausgegeben von J. Frischauf. (Ostwalds Klassiker Nr. 184.) Leipzig, Engelmann, 1911. M. 1.60